1S2 (Timoney) Tutorial sheet 18
[April 14–18, 2008]

Name: Solutions

1. For the diagonalisable matrix $A = SDS^{-1}$ where

$$S = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix}, S^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

find A and A^{10}.

Solution:

$$SD = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{5} \\ 1 & \frac{3}{10} \end{bmatrix}$$

$$SDS^{-1} = \begin{bmatrix} 1 & \frac{1}{5} \\ 1 & \frac{3}{10} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{14}{5} & -\frac{9}{5} \\ \frac{27}{10} & -\frac{17}{10} \end{bmatrix}$$

We know $A^{10} = SD^{10}S^{-1}$. So we compute

$$SD^{10} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10^{10}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{10^{10}} \\ 1 & \frac{3}{10^{10}} \end{bmatrix}$$

$$SD^{10}S^{-1} = \begin{bmatrix} 1 & \frac{2}{10^{10}} \\ 1 & \frac{3}{10^{10}} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 - \frac{2}{10^{10}} & -2 + \frac{2}{10^{10}} \\ 3 - \frac{3}{10^{10}} & -2 + \frac{3}{10^{10}} \end{bmatrix}$$

2. For the same A, find $\lim_{n\to\infty} A^n$ and e^{Ax} (with x a scalar).

Solution: Since $A^n = SD^nS^{-1}$ and

$$\lim_{n\to\infty} D^n = \lim_{n\to\infty} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10^n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

it is not hard to believe that

$$\lim_{n\to\infty} A^n = S\left(\lim_{n\to\infty} D^n\right) S^{-1} = S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$$

and we can compute

$$S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$$

A more convincing way might be to compute A^n in the same way as we computed A^{10} to get

$$A^n = \begin{bmatrix} 3 - \frac{2}{10^n} & -2 + \frac{2}{10^n} \\ 3 - \frac{3}{10^n} & -2 + \frac{3}{10^n} \end{bmatrix}$$

and then it is clear that

$$\lim_{n\to\infty} A^n = \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$$

To compute e^{Ax} we use $Ax = (SDS^{-1})x = S(Dx)S^{-1}$ (which is ok since x is a scalar variable) and so we know $e^{Ax} = Se^{Dx}S^{-1}$.

We have

$$e^{Dx} = \exp \begin{bmatrix} x & 0 \\ 0 & \frac{x}{10} \end{bmatrix} = \begin{bmatrix} e^x & 0 \\ 0 & e^{x/10} \end{bmatrix}$$

and so

$$\begin{aligned} e^{Ax} &= Se^{Dx}S^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^x & 0 \\ 0 & e^{x/10} \end{bmatrix} S^{-1} \\ &= \begin{bmatrix} e^x & 2e^{x/10} \\ e^x & 3e^{x/10} \end{bmatrix} S^{-1} = \begin{bmatrix} e^x & 2e^{x/10} \\ e^x & 3e^{x/10} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3e^x - 2e^{x/10} & -2e^x + 2e^{x/10} \\ 3e^x - 3e^{x/10} & -2e^x + 3e^{x/10} \end{bmatrix} \end{aligned}$$

3. Express the system of differential equations

$$\left\{ \begin{aligned} \frac{dy_1}{dx} &= 19y_1 - 24y_2 \\ \frac{dy_2}{dx} &= 20y_1 - 33y_2 \end{aligned} \right.$$

as a single matrix equation of the form

$$\frac{d\mathbf{y}}{dx} = A\mathbf{y}$$

What is $\mathbf{y}$? What is the matrix A?

Solution: We should take

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad A = \begin{bmatrix} 19 & -24 \\ 20 & -33 \end{bmatrix}$$

Recall then that the system of equations corresponds to the single equation for (column) matrices

$$\begin{bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{bmatrix} = \begin{bmatrix} 19y_1 - 24y_2 \\ 20y_1 - 33y_2 \end{bmatrix}$$

and this can be written

$$\frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 19 & -24 \\ 20 & -33 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

which is

$$\frac{d\mathbf{y}}{dx} = A\mathbf{y}$$

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