1S2 (Timoney) Tutorial sheet 17
[April 7–11, 2008]

Name: Solutions

1. Find the eigenvalues for the matrix

$$A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

Solution: The eigenvalues are the solutions of the characteristic equation $\det(A-\lambda I_2) = 0$. So we work out

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix} = (5-\lambda)^2 - 1$$

Solving $(5-\lambda)^2 - 1$ we get $(\lambda - 5)^2 = 1$ or $\lambda - 5 = \pm 1$. The solutions are then $\lambda = 6$ and $\lambda = 4$.

2. For the same A, find unit eigenvectors (*i.e.* length 1 eigenvectors) for each eigenvalue.

Solution: For each of the eigenvalues λ we have to solve for an eigenvector by solving $(A - \lambda I_2)\mathbf{v} = \mathbf{0}$ or row reducing $[A - \lambda I_2 : \mathbf{0}]$.

For $\lambda = 6$ we row reduce

$$\left[\begin{array}{ccc} 5-\lambda & 1 & :0 \\ 1 & 5-\lambda & :0 \end{array}\right] = \left[\begin{array}{ccc} -1 & 1 & :0 \\ 1 & -1 & :0 \end{array}\right]$$

Multiply first row by -1.

$$\left[\begin{array}{ccc} 1 & -1 & :0 \\ 1 & -1 & :0 \end{array}\right]$$

Subtract first row from second:

$$\left[\begin{array}{ccc} 1 & -1 & :0 \\ 0 & 0 & :0 \end{array}\right]$$

So we are left with one equation $v_1 - v_2 = 0$ and v_2 free. If we take $v_2 = 1$ we get a nonzero eigenvector with $v_1 = 1$ also, which is $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

To get a normalised one we take

$$\frac{1}{\|\mathbf{i}+\mathbf{j}\|}(\mathbf{i}+\mathbf{j}) = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

For $\lambda = 4$ we row reduce

$$\left[\begin{array}{ccc} 5-\lambda & 1 & :0 \\ 1 & 5-\lambda & :0 \end{array}\right] = \left[\begin{array}{ccc} 1 & 1 & :0 \\ 1 & 1 & :0 \end{array}\right]$$

Subtract first row from second:
$$\left[\begin{array}{cc|c} 1 & 1 & :0 \\ 0 & 0 & :0 \end{array}\right]$$

So we are left with one equation $v_1 + v_2 = 0$ and v_2 free. If we take $v_2 = 1$ we get a nonzero eigenvector with $v_1 = -1$, which is $\mathbf{v} = -\mathbf{i} + \mathbf{j}$.

To get a normalised one we take

$$\frac{1}{\|-\mathbf{i}+\mathbf{j}\|}(-\mathbf{i}+\mathbf{j}) = \frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

3. For the same A, find an orthogonal matrix P and a diagonal matrix D so that $A = P^t D P$.

   Solution: We should take the rows of P (or the columns of P^t) to be the two normalised eigenvectors and D to the the diagonal matrix with the eigenvalues on the diagonal (in the same order).
$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$$

   (So the first row of P is the eigenvector for the eigenvalue 6, while the second row of P is for the eigenvalue 4. It also works to take the eigenvalues in the other order so that the answer
$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$$
   is also correct. Apart from that it is possible to multiply either or both of the eigenvectors by -1 and still have normalised eigenvectors. This would give different P, but still correct as long as the order of the rows of P agrees with the order of the eigenvalues.)

4. Find the eigenvalues of the matrix

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

   Solution: We should solve the characteristic equation $\det(B - \lambda I_3) = 0$. That determinant is

$$\begin{aligned}
\det(B - \lambda I_3) &= \det\left(\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\
&= \det \begin{bmatrix} 1-\lambda & 2 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 1 & 4-\lambda \end{bmatrix} \\
&= (1-\lambda)\det\begin{bmatrix} 3-\lambda & 0 \\ 1 & 4-\lambda \end{bmatrix} - 2\det\begin{bmatrix} 1 & 0 \\ 0 & 4-\lambda \end{bmatrix} + 0 \\
&= (1-\lambda)(3-\lambda)(4-\lambda) - 2((4-\lambda) \\
&= (4-\lambda)((1-\lambda)(3-\lambda) - 2) \\
&= -(\lambda-4)((\lambda-1)(\lambda-3) - 2) \\
&= -(\lambda-4)(\lambda^2 - 4\lambda + 3 - 2) \\
&= -(\lambda-4)(\lambda^2 - 4\lambda + 1)
\end{aligned}$$

The roots of this are $\lambda = 4$ and the roots of the quadratic, which are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

So the eigenvalues are

$$4, 2+\sqrt{3} \text{ and } 2-\sqrt{3}.$$

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