1S2 (Timoney) Tutorial sheet 16
[March 31 – April 4, 2008]

Name: Solutions

1. Show matrix
$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(\pi/3) & \sin(\pi/3) \\ 0 & \sin(\pi/3) & -\cos(\pi/3) \end{bmatrix}$$
is a rotation matrix. [Hint: Is it orthogonal? What is its determinant?]

Solution: Rotation matrices are exactly orthogonal matrices of determinant 1. We can see
$$\begin{aligned} RR^t &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(\pi/3) & \sin(\pi/3) \\ 0 & \sin(\pi/3) & -\cos(\pi/3) \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(\pi/3) & \sin(\pi/3) \\ 0 & \sin(\pi/3) & -\cos(\pi/3) \end{bmatrix} \\ &= \begin{bmatrix} (-1)^1 & 0 & 0 \\ 0 & \cos^2(\pi/3) + \sin^2(\pi/3) & \cos(\pi/3)\sin(\pi/3) - \sin(\pi/3)\cos(\pi/3) \\ 0 & \sin(\pi/3)\cos(\pi/3) - \cos(\pi/3)\sin(\pi/3) & \sin^2(\pi/3) + \cos^2(\pi/3) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \end{aligned}$$
So R is orthogonal.

$$\det R = (-1)\det \begin{bmatrix} \cos(\pi/3) & \sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \end{bmatrix} = (-1)(-\cos^2(\pi/3) - \sin^2(\pi/3)) = (-1)(-1) = 1$$

So R is a rotation.

2. For the same R, find the angle θ for the rotation (up to an ambiguity between θ and $2\pi - \theta$). [Hint: use the trace.]

Solution: We know that the angle θ of rotation must satisfy
$$\mathrm{trace}(R) = 1 + 2\cos\theta$$
But
$$\mathrm{trace}(R) = -1 + \cos(\pi/3) - \cos(\pi/3) = -1$$
and so we get $1 + 2\cos\theta = -1$, $2\cos\theta = -2$, $\cos\theta = -1$.

That means $\theta = \pi$ (and in fact there is no ambiguity in this case since $2\pi - \pi = \pi$ again).

3. For the same R, find a (nonzero) vector parallel to the axis of rotation. [Hint: vector fixed by R. For this it helps to remember $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$.]

   Solution: The vector $\mathbf{u}$ we want has to satisfy $R\mathbf{u} = \mathbf{u}$ or $(R - I_3)\mathbf{u} = \mathbf{0}$.

   We have

$$R - I_3 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & \cos(\pi/3) - 1 & \sin(\pi/3) \\ 0 & \sin(\pi/3) & -\cos(\pi/3) - 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -3/2 \end{bmatrix}$$

   and so we should row reduce the augmented matrix

$$\left[\begin{array}{rrrr} -2 & 0 & 0 & :0 \\ 0 & -1/2 & \sqrt{3}/2 & :0 \\ 0 & \sqrt{3}/2 & -3/2 & :0 \end{array}\right]$$

   First divide row 1 by -2

$$\left[\begin{array}{rrrr} 1 & 0 & 0 & :0 \\ 0 & -1/2 & \sqrt{3}/2 & :0 \\ 0 & \sqrt{3}/2 & -3/2 & :0 \end{array}\right]$$

   Now multiply row 2 by -2

$$\left[\begin{array}{rrrr} 1 & 0 & 0 & :0 \\ 0 & 1 & -\sqrt{3} & :0 \\ 0 & \sqrt{3}/2 & -3/2 & :0 \end{array}\right]$$

   Subtract $\sqrt{3}/2$ times row 2 from row 3

$$\left[\begin{array}{rrrr} 1 & 0 & 0 & :0 \\ 0 & 1 & -\sqrt{3} & :0 \\ 0 & 0 & 0 & :0 \end{array}\right]$$

   So the components of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ must satisify $u_1 = 0$ and $u_2 - \sqrt{3}u_3 = 0$. (u_3 is the free variable.) Taking $u_3 = 1$ we get $u_2 = \sqrt{3}$ and $\mathbf{u} = \sqrt{3}\mathbf{j} + \mathbf{k}$.

   (That is not a unit vector. If we wanted a unit vector we should take $(\sqrt{3}\mathbf{j}+\mathbf{k})/\|\sqrt{3}\mathbf{j}+\mathbf{k}\| = (\sqrt{3}\mathbf{j} + \mathbf{k})/\sqrt{4} = (1/2)(\sqrt{3}\mathbf{j} + \mathbf{k})$.)

4. **(for homework)** Use the Gram-Schmidt method on the vectors $\mathbf{u} = \frac{1}{\sqrt{6}}(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, $\mathbf{r} = \mathbf{i}$ and $\mathbf{s} = \mathbf{j}$ (to find an othonormal basis including $\mathbf{u}$). [As you may not have time to finish this, please do it at home later.]

   Solution:

   Step 1: $\mathbf{u}$ is a unit vector already and so this step is not needed.

   Just to check

$$\|\mathbf{u}\| = \frac{1}{\sqrt{6}}\|\mathbf{i} - 2\mathbf{j} + \mathbf{k}\|\frac{1}{\sqrt{6}}\sqrt{1^2 + (-2)^2 + 1^2} = 1$$

Step 2: Calculate

$$\begin{aligned}
\mathbf{r} - (\mathbf{r}\cdot\mathbf{u})\mathbf{u} &= \mathbf{i} - \left(\frac{1}{\sqrt{6}}\right)\mathbf{u} \\
&= \mathbf{i} - \left(\frac{1}{6}\right)(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \\
&= \frac{5}{6}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{1}{6}\mathbf{k} \\
&= \frac{1}{6}(5\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\
\|\mathbf{r} - (\mathbf{r}\cdot\mathbf{u})\mathbf{u}\| &= \frac{1}{6}\|5\mathbf{i} + 2\mathbf{j} - \mathbf{k}\| \\
&= \frac{1}{6}\sqrt{5^2 + 2^2 + (-1)^2} \\
&= \frac{\sqrt{30}}{6} \\
\mathbf{v} &= \frac{\mathbf{r} - (\mathbf{r}\cdot\mathbf{u})\mathbf{u}}{\|\mathbf{r} - (\mathbf{r}\cdot\mathbf{u})\mathbf{u}\|} \\
&= \frac{1}{\sqrt{30}}(5\mathbf{i} + 2\mathbf{j} - \mathbf{k})
\end{aligned}$$

Step 3: Next compute

$$\begin{aligned}
\mathbf{s} - (\mathbf{s}\cdot\mathbf{u})\mathbf{u} - (\mathbf{s}\cdot\mathbf{v})\mathbf{v} &= \mathbf{j} - \left(\frac{-2}{\sqrt{6}}\right)\mathbf{u} - \left(\frac{2}{\sqrt{30}}\right)\mathbf{v} \\
&= \mathbf{j} - \left(\frac{-2}{6}\right)(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - \left(\frac{2}{30}\right)(5\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\
&= \mathbf{j} + \left(\frac{1}{3}\right)(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - \left(\frac{1}{15}\right)(5\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\
&= \left(\frac{1}{3} - \frac{1}{3}\right)\mathbf{i} + \left(1 - \frac{2}{3} - \frac{2}{15}\right)\mathbf{j} + \left(\frac{1}{3} + \frac{1}{15}\right)\mathbf{k} \\
&= \frac{1}{5}\mathbf{j} + \frac{2}{5}\mathbf{k} \\
&= \frac{1}{5}(\mathbf{j} + 2\mathbf{k})
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{s} - (\mathbf{s}\cdot\mathbf{u})\mathbf{u} - (\mathbf{s}\cdot\mathbf{v})\mathbf{v}\| &= \frac{1}{5}\|\mathbf{j} + 2\mathbf{k}\| \\
&= \frac{\sqrt{1+4}}{5} = \frac{\sqrt{5}}{5} \\
\mathbf{w} &= \frac{\mathbf{s} - (\mathbf{s}\cdot\mathbf{u})\mathbf{u} - (\mathbf{s}\cdot\mathbf{v})\mathbf{v}}{\|\mathbf{s} - (\mathbf{s}\cdot\mathbf{u})\mathbf{u} - (\mathbf{s}\cdot\mathbf{v})\mathbf{v}\|} \\
&= \frac{1}{\sqrt{5}}(\mathbf{j} + 2\mathbf{k})
\end{aligned}$$

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