1S2 (Timoney) Tutorial sheet 15
[March 4 – 8, 2008]

Name: Solutions

In this sheet, we consider 3 orthonormal vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ in $\mathbb{R}^3$. Let

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}, \quad Q = \begin{bmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

1. Show that the rotation matrix

$$R = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} P$$

is an orthogonal matrix.

Solution: The transpose of R is

$$R^t = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}^t (P^t)^t = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} P$$

(using the rule that the transpose of a product is the product of the transposes taken in reverse order). So

$$\begin{aligned} RR^t &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} PP^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} P \\ &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} P \\ &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P \\ &= P^t I_3 P = P^t P = I_3 \end{aligned}$$

since we know P is an orthogonal matrix.

This shows $RR^t = I_3$ and shows that $R^{-1} = R^t$.

2. For the same R, show that $\det(R) = 1$.

Solution: Since P is orthogonal we know $\det(P) = \pm 1$. We also know that the determinant of a product is the product of the determinants. So

$$\begin{aligned}
\det(R) &= \det(P^t)\det\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\det(P)\\
&= \det(P)\det\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\det(P)\\
&\quad \text{using } \det(P^t) = \det(P) \text{ and cofactor expansion}\\
&= \det(P)^2(\cos^2\theta + \sin^2\theta) = (\pm 1)^2(1) = 1
\end{aligned}$$

3. For the same R, show that

$$R^{-1} = P^t\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(-\theta) & -\sin(-\theta)\\ 0 & \sin(-\theta) & \cos(-\theta)\end{bmatrix}P$$

Solution: This makes sense geometrically as the matrix on the right is the rotation through minus the angle for R.

Algebraicly we can work it out by observing $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$ and so the matrix we have on the right is the same as

$$P^t\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}P$$

which coincides with R^t (see first question). And $R^t = R^{-1}$.

4. Show that $\det(Q) = -\det(P) = \pm 1$ and that $\det(P) = 1$ implies $\mathbf{v}\times\mathbf{w} = \mathbf{u}$. [Hint for the second part: look at the scalar triple product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ first geometrically, and then as a determinant.]

Solution: Since Q is the result of swopping two rows of P, $\det(Q) = -\det(P)$.

Since P is orthogonal, $\det(P) = \pm 1$.

Since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are orthonormal, we know that $\mathbf{v}\times\mathbf{w}$ is a vector in either the same or the opposite direction to $\mathbf{u}$. (Recall $\mathbf{v}\times\mathbf{w}$ is perpendicular to the plane of $\mathbf{v}$ and $\mathbf{w}$, but so also is $\mathbf{u}$.) In fact as the length of $\mathbf{v}\times\mathbf{w}$ is

$$\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin(\pi/2) = 1 = \|\mathbf{u}\|$$

we must have $\mathbf{v}\times\mathbf{w} = \pm\mathbf{u}$. So

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{u}\cdot(\pm\mathbf{u}) = \pm\mathbf{u}\cdot\mathbf{u} = \pm 1$$

But we know

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \det P$$

So $\det(P) = 1$ in the case $\mathbf{v} \times \mathbf{w} = \mathbf{u}$, while $\det(P) = -1$ in the case $\mathbf{v} \times \mathbf{w} = -\mathbf{u}$.

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