1S2 (Timoney) Tutorial sheet 14
[February 25 – 28, 2008]

Name: Solutions.

1. Convert $\frac{23}{7}$ to binary.

   Solution: First $\frac{23}{7} = 3 + \frac{2}{7}$ and $3 = (11)_2$. We concentrate on the fractional part $\frac{2}{7}$. Imagine the binary expansion as

$$
\begin{aligned}
\frac{2}{7} &= (0.b_1b_2b_3\ldots)_2 \\
\text{Double} & \\
\frac{4}{7} &= (b_1.b_2b_3b_4\ldots)_2 \\
\text{Integer parts} & \\
& \boxed{b_1 = 0} \\
\text{Fractional parts} & \\
\frac{4}{7} &= (0.b_2b_3b_4\ldots)_2 \\
\text{Double} & \\
\frac{8}{7} &= (b_2.b_3b_4b_5\ldots)_2 \\
\text{Integer parts} & \\
& \boxed{b_2 = 1} \\
\text{Fractional parts} & \\
\frac{1}{7} &= (0.b_3b_4b_5\ldots)_2 \\
\text{Double} & \\
\frac{2}{7} &= (b_3.b_4b_5b_6\ldots)_2 \\
\text{Integer parts} & \\
& \boxed{b_3 = 0} \\
\text{Fractional parts} & \\
\frac{2}{7} &= (0.b_4b_5b_6\ldots)_2 \\
&= (0.b_1b_2b_3\ldots)_2
\end{aligned}
$$

   Thus the pattern repeats, $b_4 = b_1$, $b_5 = b_2$, etc and so $\frac{2}{7} = (0.\overline{010})_2$. The answer is

$$\frac{23}{7} = 3 + \frac{2}{7} = (11.\overline{010})_2$$

2. Is this an orthogonal matrix?

$$A = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

[Hint: Work out AA^t.]

Solution:

$$\begin{aligned} AA^t &= \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{8}+\frac{1}{2}+\frac{3}{8} & \frac{1}{8}-\frac{1}{2}-\frac{3}{8} & \frac{\sqrt{3}}{4\sqrt{2}}+0-\frac{\sqrt{3}}{4\sqrt{2}} \\ \frac{1}{8}-\frac{1}{2}+\frac{3}{8} & \frac{1}{8}+\frac{1}{2}\frac{3}{8} & \frac{\sqrt{3}}{4\sqrt{2}}+0-\frac{\sqrt{3}}{4\sqrt{2}} \\ \frac{\sqrt{3}}{4\sqrt{2}}+0-\frac{\sqrt{3}}{4\sqrt{2}} & \frac{\sqrt{3}}{4\sqrt{2}}+0-\frac{\sqrt{3}}{4\sqrt{2}} & \frac{3}{4}+0+\frac{1}{4} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \end{aligned}$$

So (we know $(A^t)A = I_3$ automatically and) $A^{-1} = A^t$. Thus A is orthogonal.

3. Show that if A and B are orthogonal $n \times n$ matrices, then AB is also orthogonal.

Solution: We know $(AB)^{-1} = B^{-1}A^{-1}$ (because A and B are invertible). But $B^{-1} = B^t$ and $A^{-1} = A^t$. So we have

$$(AB)^{-1} = B^{-1}A^{-1} = B^tA^t = (AB)^t$$

so that AB is orthogonal.

4. Show that if A is an orthgonal matrix, then $\det(A) = \pm 1$.

Solution: If A is orthgonal, $AA^t = I_n$. So $\det(AA^t) = \det(I_n) = 1$. So

$$\det(A)\det(A^t) = 1.$$

But $\det(A^t) = \det(A)$. Thus we get

$$(\det(A))^2 = 1$$

and that tells us $\det(A)$ has to be either 1 or -1.

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