1S2 (Timoney) Tutorial sheet 11
[February 4 – 8, 2008]

Name: Solutions

1. Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 11\mathbf{j} + 14\mathbf{k}$.

   Solution:

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 8 \\ 1 & 11 & 14 \end{bmatrix} \\
&= \mathbf{i} \det \begin{bmatrix} 4 & 8 \\ 11 & 14 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} -3 & 8 \\ 1 & 14 \end{bmatrix} + \mathbf{k} \det \begin{bmatrix} -3 & 4 \\ 1 & 11 \end{bmatrix} \\
&= (56 - 88)\mathbf{i} - (-42 - 8)\mathbf{j} + (-33 - 4)\mathbf{k} \\
&= -32\mathbf{i} + 50\mathbf{j} - 37\mathbf{k}
\end{aligned}$$

2. For the same vectors $\mathbf{v}$ and $\mathbf{w}$, find $\mathbf{w} \times \mathbf{v}$, $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{v})$ and $(3\mathbf{v} + 4\mathbf{w}) \times (\mathbf{v} - \mathbf{w})$.

   Solution:

$$\begin{aligned}
\mathbf{w} \times \mathbf{v} &= -\mathbf{v} \times \mathbf{w} \\
&= 32\mathbf{i} - 50\mathbf{j} + 37\mathbf{k} \\
\mathbf{v} \cdot (\mathbf{w} \times \mathbf{v}) &= 0 \\
& \quad (\text{since } \mathbf{v} \text{ always} \perp \mathbf{w} \times \mathbf{v}) \\
(3\mathbf{v} + 4\mathbf{w}) \times (\mathbf{v} - \mathbf{w}) &= 3(\mathbf{v} \times \mathbf{v}) - 3(\mathbf{v} \times \mathbf{w}) \\
& \quad +4(\mathbf{w} \times \mathbf{v}) - 4(\mathbf{w} \times \mathbf{w}) \\
&= \mathbf{0} - 3(\mathbf{v} \times \mathbf{w}) - 4(\mathbf{v} \times \mathbf{w}) + \mathbf{0} \\
&= -7(\mathbf{v} \times \mathbf{w}) \\
&= -224\mathbf{i} + 350\mathbf{j} - 259\mathbf{k}
\end{aligned}$$

3. Use cross products to find a normal vector to the plane in $\mathbb{R}^3$ that contains the 3 points $(1, 4, 6)$, $(2, -1, 5)$ and $(7, -8, 2)$.

   Solution: The vectors $2\mathbf{i} - \mathbf{j} + 5\mathbf{k} - (\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}) = \mathbf{i} - 5\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - 12\mathbf{j} - 4\mathbf{k} - (\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}) = 6\mathbf{i} - 12\mathbf{j} - 4\mathbf{k}$ must lie in the plane. Thus their cross product is normal to the plane. That cross product is

$$\begin{aligned}
\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -5 & -1 \\ 6 & -12 & -4 \end{bmatrix} &= \mathbf{i} \det \begin{bmatrix} 5 & -1 \\ -12 & -4 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix} + \mathbf{k} \det \begin{bmatrix} 1 & -5 \\ 6 & -12 \end{bmatrix} \\
&= (-20 - 12)\mathbf{i} - (-4 + 6)\mathbf{j} + (-12 + 30)\mathbf{k} \\
&= -32\mathbf{i} - 2\mathbf{j} + 18\mathbf{k}
\end{aligned}$$

4. Use the determinant method to find the equation of the circle in $\mathbb{R}^2$ that passes through the 3 points $(4,5)$, $(7,11)$ and $(5,1)$.

Solution: The equation is

$$\det \begin{bmatrix} x^2+y^2 & x & y & 1 \\ 4^2+5^2 & 4 & 5 & 1 \\ 7^2+11^2 & 7 & 11 & 1 \\ 5^2+1^2 & 5 & 1 & 1 \end{bmatrix} = 0$$

$$\begin{aligned}
\det \begin{bmatrix} x^2+y^2 & x & y & 1 \\ 41 & 4 & 5 & 1 \\ 170 & 7 & 11 & 1 \\ 26 & 5 & 1 & 1 \end{bmatrix} &= (x^2+y^2)\det \begin{bmatrix} 4 & 5 & 1 \\ 7 & 11 & 1 \\ 5 & 1 & 1 \end{bmatrix} - x\det \begin{bmatrix} 41 & 5 & 1 \\ 170 & 11 & 1 \\ 26 & 1 & 1 \end{bmatrix} \\
&\quad + y\det \begin{bmatrix} 41 & 4 & 1 \\ 170 & 7 & 1 \\ 26 & 5 & 1 \end{bmatrix} - \det \begin{bmatrix} 41 & 4 & 5 \\ 170 & 7 & 11 \\ 26 & 5 & 1 \end{bmatrix} \\
\det \begin{bmatrix} 4 & 5 & 1 \\ 7 & 11 & 1 \\ 5 & 1 & 1 \end{bmatrix} &= \det \begin{bmatrix} 4 & 5 & 1 \\ 3 & 6 & 0 \\ 1 & -4 & 0 \end{bmatrix} \\
&= 4(0) - 5(0) + 1(-12-6) = -18 \\
\det \begin{bmatrix} 41 & 5 & 1 \\ 170 & 11 & 1 \\ 26 & 1 & 1 \end{bmatrix} &= \det \begin{bmatrix} 41 & 5 & 1 \\ 129 & 6 & 0 \\ -15 & -4 & 0 \end{bmatrix} \\
&= 41(0) - 5(0) + 1(-516+90) = -426 \\
\det \begin{bmatrix} 41 & 4 & 1 \\ 170 & 7 & 1 \\ 26 & 5 & 1 \end{bmatrix} &= \det \begin{bmatrix} 41 & 4 & 1 \\ 129 & 3 & 0 \\ -15 & 1 & 0 \end{bmatrix} \\
&= 41(0) - 4(0) + 1(129+45) = 174 \\
\det \begin{bmatrix} 41 & 4 & 5 \\ 170 & 7 & 11 \\ 26 & 5 & 1 \end{bmatrix} &= 41(7-55) - 4(170-286) + 5(850-182) = 1836
\end{aligned}$$

Equation is

$$-18(x^2+y^2) + 426x + 174y - 1836 = 0.$$

Richard M. Timoney