1S2 (Timoney) sample for part 2
of 1S1/1S2 christmas test

The instructions will say to do 3 questions from this part. They will also state:

Log tables are available from the invigilators, if required.
Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. Use Gauss-Jordan elimination to describe all solutions of the following system of linear equations:

$$\begin{array}{rcrcrcrcr} 5x_1 & - & x_2 & - & x_3 & - & 8x_4 & = & 5 \\ 15x_1 & + & 2x_2 & + & 3x_3 & + & 5x_4 & = & 10 \\ 10x_1 & - & 2x_2 & + & 4x_3 & + & 2x_4 & = & 8 \end{array}$$

Solution: First we represent the system by an augmented matrix.

$$\left[\begin{array}{rrrrcr} 5 & -1 & -1 & -8 & : & 5 \\ 15 & 2 & 3 & 5 & : & 10 \\ 10 & -2 & 4 & 2 & : & 8 \end{array}\right]$$

Now we start by Gausssian elimination

$$\left[\begin{array}{rrrrcr} 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{8}{5} & : & 1 \\ 15 & 2 & 3 & 5 & : & 10 \\ 10 & -2 & 4 & 2 & : & 8 \end{array}\right| \begin{array}{l} \text{OldRow1} \times \frac{1}{5} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{rrrrcr} 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{8}{5} & : & 1 \\ 0 & 5 & 6 & 29 & : & -5 \\ 0 & 0 & 6 & 18 & : & -2 \end{array}\right| \begin{array}{l} \\ \text{OldRow2} - 15 \times \text{OldRow1} \\ \text{OldRow2} - 10 \times \text{OldRow1} \end{array}$$

$$\left[\begin{array}{rrrrcr} 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{8}{5} & : & 1 \\ 0 & 1 & \frac{6}{5} & \frac{29}{5} & : & -1 \\ 0 & 0 & 6 & 18 & : & -2 \end{array}\right| \begin{array}{l} \\ \text{OldRow2} \times \frac{1}{5} \\ \\ \end{array}$$

$$\left[\begin{array}{rrrrcr} 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{8}{5} & : & 1 \\ 0 & 1 & \frac{6}{5} & \frac{29}{5} & : & -1 \\ 0 & 0 & 1 & 3 & : & -\frac{1}{3} \end{array}\right| \begin{array}{l} \\ \\ \text{OldRow3} \times \frac{1}{6} \end{array}$$

This finishes Gaussian elimination (and now the augmented matrix is in row echelon form). As we are doing Gauss-Jordan we continue as follows

$$\left[\begin{array}{rrrrcr} 1 & -\frac{1}{5} & 0 & -1 & : & \frac{14}{15} \\ 0 & 1 & 0 & \frac{11}{5} & : & -\frac{3}{5} \\ 0 & 0 & 1 & 3 & : & -\frac{1}{3} \end{array}\right| \begin{array}{l} \text{OldRow1} + \frac{1}{5} \times \text{OldRow3} \\ \text{OldRow2} - \frac{6}{5} \times \text{OldRow3} \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{14}{25} & : & \frac{61}{75} \\ 0 & 1 & 0 & \frac{11}{5} & : & -\frac{3}{5} \\ 0 & 0 & 1 & 3 & : & -\frac{1}{3} \end{array}\right| \text{OldRow1} + \tfrac{1}{5} \times \text{OldRow2}$$

This is now in reduced row-echelon form and we have completed the Gauss-Jordan method. We write equations for this augmented matrix

$$\left\{\begin{array}{lllllll} x_1 & & & - & \frac{14}{25}x_4 & = & \frac{61}{75} \\ & x_2 & & + & \frac{11}{5}x_4 & = & -\frac{3}{5} \\ & & x_3 & + & 3x_4 & = & -\frac{1}{3} \end{array}\right.$$

and now we solve for the variables occuriing on the left of each of these to get

$$\left\{\begin{array}{lll} x_1 & = & \frac{61}{75} + \frac{14}{25}x_4 \\ x_2 & = & -\frac{3}{5} - \frac{11}{5}x_4 \\ x_3 & = & -\frac{1}{3} - 3x_4 \\ x_4 & & \text{free} \end{array}\right.$$

(We have solved for x_1, x_2 and x_3 in terms of x_4. Every value of the free variable x_4 gives a solution and every solution arises in this way. So this is a complete description of all the solutions.)

2. (a) For $\mathbf{v} = -3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = 6\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, calculate $\|5\mathbf{v} - \mathbf{w}\|$ and the projection $\mathrm{proj}_{\mathbf{w}}(\mathbf{v})$ of $\mathbf{v}$ along the direction of $\mathbf{w}$.

Solution:

$$\begin{aligned} 5\mathbf{v} &= -15\mathbf{i} + 35\mathbf{j} + 10\mathbf{k} \\ 5\mathbf{v} - \mathbf{w} &= -21\mathbf{i} + 38\mathbf{j} + 5\mathbf{k} \\ \|5\mathbf{v} - \mathbf{w}\| &= \sqrt{(-21)^2 + 38^2 + 5^2} \\ &= \sqrt{1910} \\ \mathrm{proj}_{\mathbf{w}}(\mathbf{v}) &= \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} \\ &= \frac{(-3)(6) + 7(-3) + 2(5)}{6^2 + (-3062 + 5^2}\mathbf{w} \\ &= \frac{-29}{70}(6\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) \\ &= \frac{-87}{35}\mathbf{i} + \frac{87}{70}\mathbf{j} - \frac{29}{14}\mathbf{k} \end{aligned}$$

(b) Find both the parametric and cartesian equations for the line in space which passes through the point $(1, 2, 3)$ and is perpendicular to the plane

$$5x - 6y + z = 4.$$

Solution: A normal vector to the plane is $5\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ and this gives us a vector parallel to the line. So we can write the parametric equations for the line in vector form as

$$\mathbf{x} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(5\mathbf{i} - 6\mathbf{j} + \mathbf{k})$$

or as the 3 scalar equations

$$\begin{aligned}
x &= 1+5t \\
y &= 2-6t \\
z &= 3+t.
\end{aligned}$$

Solving each of these for t we get

$$\frac{x-1}{5}=t, \quad \frac{y-2}{6}=t, \quad \frac{z-3}{1}=t$$

Thus the cartesian equations are

$$\frac{x-1}{5}=\frac{y-2}{6}=\frac{z-3}{1}$$

3. (a) Let $\mathbf{x}=(2,1,-3,5,2)$ and $\mathbf{y}=(3,0,3,-4,-2)$ (in $\mathbb{R}^5$). Compute the cosine of the angle between $\mathbf{x}$ and $\mathbf{y}$.

Solution: We know $\mathbf{x}\cdot\mathbf{y}=\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$ or

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \\
&= \frac{2(3)+(1)(0)+(-3)(3)+5(-4)+2(-2)}{\sqrt{2^2+1^1+(-3)^2+5^2+2^2}\sqrt{3^2+0+3^2+(-4)^2+(-2)^2}} \\
&= \frac{-27}{\sqrt{43}\sqrt{38}}
\end{aligned}$$

(b) For

$$a=\begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -2 \\ 3 & 1 & 5 \\ -4 & 3 & 2 \end{bmatrix}, b=\begin{bmatrix} 2 & -2 & 1 & 4 \\ 4 & -5 & 2 & 3 \\ 5 & 7 & 1 & 4 \end{bmatrix}, c=\begin{bmatrix} \frac{1}{2} & 3 \\ 2 & -1 \\ 4 & -2 \\ 1 & 0 \end{bmatrix}$$

compute ab, ba and bc.

Solution:

$$ab=\begin{bmatrix} 15 & 22 & 3 & 17 \\ -2 & -22 & 2 & 8 \\ 35 & 24 & 10 & 35 \\ 14 & 7 & 4 & 1 \end{bmatrix}$$

$$ba=\begin{bmatrix} -17 & 11 & 23 \\ -18 & 7 & 38 \\ 25 & 8 & 14 \end{bmatrix}$$

$$bc=\begin{bmatrix} 28 & 6 \\ 49 & 13 \\ 82 & 6 \end{bmatrix}$$

4. (a) What output would be produced by the following `Mathematica` instruction, and what does it mean?

```
FactorInteger[36]
```

Solution: The output is

```
{{2, 2}, {3, 2}}
```

and it means that the prime factors of 36 are 2 and 3, and that each divides 36 twice. So the factorisation of 36 into primes (or powers of distinct primes) is

$$36 = 2^2 3^2$$

(b) Write a `Mathematica` instruction to factor $3x^2 + 2x - 1$.

Solution:

```
Factor[3x^2 + 2x -1]
```

(c) Write a `Mathematica` instruction to graph $y = \sin(x^2 + 1)$ for x in the range $-3 \leq x \leq 3$.

Solution:

```
Plot[ Sin[x^2 + 1], {x, -3, 3}]
```

(d) What does the following `Mathematica` instruction mean?

```
Solve[x^2 + 4 y x - 5 y^2 == 0, x]
```

Work out mathematically, in as much detail as you can, what the result will be (and give reasons).

Solution: It mens to solve the equation $x^2 + 4yx - 5y^2 = 0$ for x (in terms of any other quantities that occur, in this case in terms of y).

As this quadradic in x can be factored

$$x^2 + 4yx - 5y^2 = (x + 5y)(x - y)$$

the solutions of $x^2 + 4yx - 5y^2 = 0$ are $x = -5y$ and $x = y$ and so the output will say that. In fact the output will be

```
{{x -> -5 y}, {x -> y}}
```

(e) The following shows a portion of a spreadsheet.

	A	B	C	D
1	23			
2	19			
3	13			
4	10			
5	-5			
6				
7				
8				
9				
10				
11				

If you type into cell **B3** the keystrokes `=2*A4+1` and return, what will then show in cell **B3**? If you then copy from cell **B3** and paste into cell **B4**, what will show in cell **B4** afterwards?

Solution: The number in cell **A4** is 10 and so entering the formula `=2*A4+1` will result in the calculation of $2 \times 10 + 1$. The cell **B3** will then show 21.

If you copy the formula `=2*A4+1` from cell **B3** and paste it into cell **B4**, it will become the formula `=2*A5+1`. So it will compute $2 \times (-5) + 1$ and the cell **B4** will show -9.