1S2 Solutions: May–June 2008

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1. (a) With the aid of a table like the following, show how the integer 37 would be converted to a bit pattern (zeros and ones) in a computer with 32 bit integers.

37			...						
Bit position:	1	2	...	27	28	29	30	31	32

Solution: We know $37 = 32 + 5 = 2^5 + 2^2 + 1 = (100101)_2$.

37	0	0	0...0	1	0	0	1	0	1
Bit position:	1	2	...	27	28	29	30	31	32

(b) In a different context a computer will store 37 as a (single precision) floating point number with a (binary) mantissa and exponent. Find the appropriate mantissa and exponent. Then explain how this would convert to a bit pattern by use of a table like the one above.

Solution: $32 = (100101)_2 = (1.00101)_2 \times 2^5$ and so the mantissa is $(1.00101)_2$ and the exponent is $5 = (101)_2$.

0	0	0	0	0	0	1	0	1	1	0	0	1	0	1	0	0	...	0	0
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	...	31	32
±				exponent									mantissa less sign						

(c) Convert the octal number $(27214)_8$ to hexadecimal by first converting it to binary using the "3 binary for one octal" rule, and then using a similar rule to convert to hexadecimal.

Solution:

$$(27214)_8 = (010\ 111\ 010\ 001\ 100)_2 = (10111010001100)_2$$

is the number in binary.

$$(10111010001100)_2 = (0010\ 1110\ 1000\ 1100)_2 = (2e8c)_{16}$$

(d) Convert 40/7 to binary (and indicate the pattern of the expansion).

Solution: Solution: First $\dfrac{40}{7} = 5 + \dfrac{5}{7}$ and $5 = (101)_2$. We concentrate on the fractional part $\dfrac{5}{7}$.

Imagine the binary expansion as

$$\begin{aligned}
\frac{5}{7} &= (0.b_1 b_2 b_3 \ldots)_2 \\
\text{Double} & \\
\frac{10}{7} &= (b_1.b_2 b_3 b_4 \ldots)_2 \\
\text{Integer parts} & \\
& \boxed{b_1 = 1} \\
\text{Fractional parts} & \\
\frac{3}{7} &= (0.b_2 b_3 b_4 \ldots)_2 \\
\text{Double} & \\
\frac{6}{7} &= (b_2.b_3 b_4 b_5 \ldots)_2 \\
\text{Integer parts} & \\
& \boxed{b_2 = 0} \\
\text{Fractional parts} & \\
\frac{6}{7} &= (0.b_3 b_4 b_5 \ldots)_2 \\
\text{Double} & \\
\frac{12}{7} &= (b_3.b_4 b_5 b_6 \ldots)_2 \\
\text{Integer parts} & \\
& \boxed{b_3 = 1} \\
\text{Fractional parts} & \\
\frac{5}{7} &= (0.b_4 b_5 b_6 \ldots)_2 \\
&= (0.b_1 b_2 b_3 \ldots)_2
\end{aligned}$$

Thus the pattern repeats, $b_4 = b_1$, $b_5 = b_2$, etc and so $\frac{5}{7} = (0.\overline{101})_2$.
The answer is

$$\frac{40}{7} = 5 + \frac{5}{7} = (101.\overline{101})_2$$

2. (a) For

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Find D^{-1} and D^5.

Solution:

$$D^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & -1/3 \end{bmatrix}$$

$$D^5 = \begin{bmatrix} (-1)^5 & 0 & 0 & 0 \\ 0 & 4^5 & 0 & 0 \\ 0 & 0 & 3^5 & 0 \\ 0 & 0 & 0 & (-3)^5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1024 & 0 & 0 \\ 0 & 0 & 243 & 0 \\ 0 & 0 & 0 & -243 \end{bmatrix}$$

(b) For

$$U = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

find U^{-1}.

Solution: We should use Gauss-Jordan elimination on $[U|I_4]$ (to get $[I_4|U^{-1}]$).

$$\begin{bmatrix} 1 & -2 & 3 & 0 & :1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -2 & :0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 & :0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & :0 & 0 & 0 & 1 \end{bmatrix}$$

Add $(-5)\times$ row 4 to row 3 and also add $2\times$ row 4 to row 2

$$\begin{bmatrix} 1 & -2 & 3 & 0 & :1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & :0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & :0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 & :0 & 0 & 0 & 1 \end{bmatrix}$$

Add -4 times row 3 to row 2 and -3 times row 3 to row 1

$$\begin{bmatrix} 1 & -2 & 0 & 0 & :1 & 0 & -3 & 15 \\ 0 & 1 & 0 & 0 & :0 & 1 & -4 & 22 \\ 0 & 0 & 1 & 0 & :0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 & :0 & 0 & 0 & 1 \end{bmatrix}$$

Add 2 times row 2 to row 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 & :1 & 2 & -11 & 59 \\ 0 & 1 & 0 & 0 & :0 & 1 & -4 & 22 \\ 0 & 0 & 1 & 0 & :0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 & :0 & 0 & 0 & 1 \end{bmatrix}$$

Thus

$$U^{-1} = \begin{bmatrix} 1 & 2 & -11 & 59 \\ 0 & 1 & -4 & 22 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) With the same U as in the previous part, find $L = U^t$ and L^{-1}.

Solution:

$$L = U^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & -2 & 5 & 1 \end{bmatrix}$$

$$L^{-1} = (U^t)^{-1} = (U^{-1})^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -11 & -4 & 1 & 0 \\ 59 & 22 & -5 & 1 \end{bmatrix}$$

(d) With the same D, L and U as in the earlier parts of the question, find $(LDU)^{-1}$.

$$
\begin{aligned}
(LDU)^{-1} &= U^{-1}D^{-1}L^{-1} \\
&= \begin{bmatrix} 1 & 2 & -11 & 59 \\ 0 & 1 & -4 & 22 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & -1/3 \end{bmatrix} L^{-1} \\
&= \begin{bmatrix} -1 & 1/2 & -11/3 & -59/3 \\ 0 & 1/4 & -4/3 & -22/3 \\ 0 & 0 & 1/3 & 5/3 \\ 0 & 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -11 & -4 & 1 & 0 \\ 59 & 22 & -5 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1120 & -835/2 & 284/3 & -59/3 \\ -835/2 & -623/4 & 106/3 & -22/3 \\ 284/3 & 106/3 & -8 & 5/3 \\ -59/3 & -22/3 & 5/3 & -1/3 \end{bmatrix}
\end{aligned}
$$

3. (a) For

$$
A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -4 & -2 \\ -4 & 0 & -2 \end{bmatrix}
$$

find the determinant $\det(A)$.

Solution:

$$
\begin{aligned}
\det(A) &= 2\det\begin{bmatrix} -4 & -2 \\ 0 & -2 \end{bmatrix} - (-1)\det\begin{bmatrix} 0 & -2 \\ -4 & -2 \end{bmatrix} + 0 \\
&= 2(8-0) + (0-8) = 8
\end{aligned}
$$

(b) Find the volume of the parallelopiped in space where three of the edges are of the same length as and parallel to the vectors $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = -4\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = -4\mathbf{i} - 2\mathbf{k}$.

Solution: This volume is given by the absolute value of the determinant of the matrix with the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ as its rows. But that matrix is the same A and so the volume is $|\det(A)| = |8| = 8$.

(c) Use the determinant method to find the equation of the circle in $\mathbb{R}^2$ that passes through the 3 points $(4,5)$, $(7,11)$ and $(5,1)$.

Solution: The equation is

$$\det \begin{bmatrix} x^2+y^2 & x & y & 1 \\ 4^2+5^2 & 4 & 5 & 1 \\ 7^2+11^2 & 7 & 11 & 1 \\ 5^2+1^2 & 5 & 1 & 1 \end{bmatrix} = 0$$

$$\begin{aligned}
\det \begin{bmatrix} x^2+y^2 & x & y & 1 \\ 41 & 4 & 5 & 1 \\ 170 & 7 & 11 & 1 \\ 26 & 5 & 1 & 1 \end{bmatrix} &= (x^2+y^2)\det \begin{bmatrix} 4 & 5 & 1 \\ 7 & 11 & 1 \\ 5 & 1 & 1 \end{bmatrix} - x\det \begin{bmatrix} 41 & 5 & 1 \\ 170 & 11 & 1 \\ 26 & 1 & 1 \end{bmatrix} \\
&\quad + y\det \begin{bmatrix} 41 & 4 & 1 \\ 170 & 7 & 1 \\ 26 & 5 & 1 \end{bmatrix} - \det \begin{bmatrix} 41 & 4 & 5 \\ 170 & 7 & 11 \\ 26 & 5 & 1 \end{bmatrix} \\
\det \begin{bmatrix} 4 & 5 & 1 \\ 7 & 11 & 1 \\ 5 & 1 & 1 \end{bmatrix} &= \det \begin{bmatrix} 4 & 5 & 1 \\ 3 & 6 & 0 \\ 1 & -4 & 0 \end{bmatrix} \\
&= 4(0) - 5(0) + 1(-12-6) = -18 \\
\det \begin{bmatrix} 41 & 5 & 1 \\ 170 & 11 & 1 \\ 26 & 1 & 1 \end{bmatrix} &= \det \begin{bmatrix} 41 & 5 & 1 \\ 129 & 6 & 0 \\ -15 & -4 & 0 \end{bmatrix} \\
&= 41(0) - 5(0) + 1(-516+90) = -426 \\
\det \begin{bmatrix} 41 & 4 & 1 \\ 170 & 7 & 1 \\ 26 & 5 & 1 \end{bmatrix} &= \det \begin{bmatrix} 41 & 4 & 1 \\ 129 & 3 & 0 \\ -15 & 1 & 0 \end{bmatrix} \\
&= 41(0) - 4(0) + 1(129+45) = 174 \\
\det \begin{bmatrix} 41 & 4 & 5 \\ 170 & 7 & 11 \\ 26 & 5 & 1 \end{bmatrix} &= 41(7-55) - 4(170-286) + 5(850-182) = 1836
\end{aligned}$$

So the equation is

$$-18(x^2+y^2) + 426x + 174y - 1836 = 0.$$

4. (a) Give the definition of an orthogonal matrix.

Solution: An $n \times n$ matrix A is called orthogonal if its transpose is the same as its inverse.

(b) If A and B are orthogonal $n \times n$ matrices, show that AB is orthogonal.

Solution: Since $A^t = A^{-1}$ we have $AA^t = I_n$ and similarly $BB^t = I_n$.

Now

$$(AB)(AB)^t = ABB^tA^t = A(BB^t)A^t = AI_nA^t = AA^t = I_n$$

This shows that $(AB)^{-1} = (AB)^t$ and so AB is orthogonal.

(c) Show that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

is orthogonal no matter what (real number) value θ has.

Solution:

$$\begin{aligned}
&\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}^t \\
= &\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \\
= &\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ 0 & \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} \\
= &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3
\end{aligned}$$

Therefore $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix.

(d) If P is an orthogonal 3×3 matrix, show that

$$R = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} P$$

is orthogonal and has determinant $\det(R) = 1$.

Solution: We could argue that it is a product of orthogonal matrices, or do the following.

The transpose of R is

$$R^t = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}^t (P^t)^t = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} P$$

(using the rule that the transpose of a product is the product of the transposes taken in reverse order). So

$$\begin{aligned} RR^t &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} PP^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} P \\ &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} P \\ &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P \\ &= P^t I_3 P = P^t P = I_3 \end{aligned}$$

since we know P is an orthogonal matrix.

This shows $RR^t = I_3$ and shows that $R^{-1} = R^t$.

Since P is orthogonal we know $\det(P) = \pm 1$. We also know that the determinant of a product is the product of the determinants.

So

$$\begin{aligned}
\det(R) &= \det(P^t)\det\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\det(P)\\
&= \det(P)\det\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\det(P)\\
&\quad\ \text{using } \det(P^t)=\det(P) \text{ and cofactor expansion}\\
&= \det(P)^2(\cos^2\theta+\sin^2\theta) = (\pm 1)^2(1) = 1
\end{aligned}$$

5. (a) Show that the matrix

$$R = \begin{bmatrix}-1 & 0 & 0\\ 0 & \cos(\pi/4) & \sin(\pi/4)\\ 0 & \sin(\pi/4) & -\cos(\pi/4)\end{bmatrix}$$

is a rotation matrix.

Solution: Rotation matrices are exactly orthogonal matrices of determinant 1. We can see

$$\begin{aligned}
RR^t &= \begin{bmatrix}-1 & 0 & 0\\ 0 & \cos(\pi/4) & \sin(\pi/4)\\ 0 & \sin(\pi/4) & -\cos(\pi/4)\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & \cos(\pi/4) & \sin(\pi/4)\\ 0 & \sin(\pi/4) & -\cos(\pi/4)\end{bmatrix}\\
&= \begin{bmatrix}(-1)^2 & 0 & 0\\ 0 & \cos^2\frac{\pi}{4}+\sin^2\frac{\pi}{4} & \cos\frac{\pi}{4}\sin\frac{\pi}{4}-\sin\frac{\pi}{4}\cos\frac{\pi}{4}\\ 0 & \sin\frac{\pi}{4}\cos\frac{\pi}{4}-\cos\frac{\pi}{4}\sin\frac{\pi}{4} & \sin^2\frac{\pi}{4}+\cos^2\frac{\pi}{4}\end{bmatrix}\\
&= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I_3
\end{aligned}$$

So R is orthogonal.

$$\begin{aligned}
\det R &= (-1)\det\begin{bmatrix}\cos(\pi/4) & \sin(\pi/4)\\ \sin(\pi/4) & -\cos(\pi/4)\end{bmatrix}\\
&= (-1)(-\cos^2(\pi/4)-\sin^2(\pi/4)) = (-1)(-1) = 1
\end{aligned}$$

So R is a rotation.

(b) For the same R, find the cosine of the angle θ for the rotation.

Solution: We know that the angle θ of rotation must satisfy

$$\text{trace}(R) = 1 + 2\cos\theta$$

But

$$\text{trace}(R) = -1 + \cos(\pi/4) - \cos(\pi/4) = -1$$

and so we get $1 + 2\cos\theta = -1$, $2\cos\theta = -2$, $\cos\theta = -1$.

(c) Suppose

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

is an orthogonal matrix and let

$$Q = \begin{bmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

.

Show that the rows of P are orthonormal vectors (in $\mathbb{R}^3$) and use that to compute PQ^t.

Solution:

$$PP^t = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

This product works out as

$$\begin{bmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} & \mathbf{u} \cdot \mathbf{w} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{w} \\ \mathbf{w} \cdot \mathbf{u} & \mathbf{w} \cdot \mathbf{v} & \mathbf{w} \cdot \mathbf{w} \end{bmatrix}$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$.

But we know also that

$$PP^t = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(since P is orthogonal, $P^t = P^{-1}$). Comparing we see

$$\mathbf{u}\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{v} = \mathbf{w}\cdot\mathbf{w} = 1$$

and all dot products between pairs of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are zero. That says the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. Since $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2$ we have

$$\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 = \|\mathbf{w}\|^2 = 1$$

and so the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are also unit vectors. So they are othonormal.

$$PQ^t = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} \begin{bmatrix} \mathbf{u} & \mathbf{w} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{w} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{v}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{w} & \mathbf{v}\cdot\mathbf{v} \\ \mathbf{w}\cdot\mathbf{u} & \mathbf{w}\cdot\mathbf{w} & \mathbf{w}\cdot\mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

6. For this question, take

$$A = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}$$

(a) Find the eigenvalues of A.

Solution: The eigenvalues are the solutions of the characteristic equation $\det(A - \lambda I_2) = 0$. So we work out

$$\det(A-\lambda I_2) = \det\left(\begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{bmatrix} = (6-\lambda)^2 - 1$$

Solving $(6-\lambda)^2 - 1$ we get $(\lambda - 6)^2 = 1$ or $\lambda - 6 = \pm 1$. The solutions are then $\lambda = 7$ and $\lambda = 5$.

(b) Find an eigenvector for each eigenvalue (of A).

Solution: For each of the eigenvalues λ we have to solve for an eigenvector by solving $(A-\lambda I_2)\mathbf{v} = \mathbf{0}$ or row reducing $[A-\lambda I_2 : \mathbf{0}]$. For $\lambda = 7$ we row reduce

$$\begin{bmatrix} 6-\lambda & 1 & :0 \\ 1 & 6-\lambda & :0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & :0 \\ 1 & -1 & :0 \end{bmatrix}$$

Multiply first row by -1.

$$\begin{bmatrix} 1 & -1 & :0 \\ 1 & -1 & :0 \end{bmatrix}$$

Subtract first row from second:

$$\begin{bmatrix} 1 & -1 & :0 \\ 0 & 0 & :0 \end{bmatrix}$$

So we are left with one equation $v_1 - v_2 = 0$ and v_2 free. If we take $v_2 = 1$ we get a nonzero eigenvector with $v_1 = 1$ also, which is $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

For $\lambda = 5$ we row reduce

$$\begin{bmatrix} 6-\lambda & 1 & :0 \\ 1 & 6-\lambda & :0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & :0 \\ 1 & 1 & :0 \end{bmatrix}$$

Subtract first row from second:

$$\begin{bmatrix} 1 & 1 & :0 \\ 0 & 0 & :0 \end{bmatrix}$$

So we are left with one equation $v_1 + v_2 = 0$ and v_2 free. If we take $v_2 = 1$ we get a nonzero eigenvector with $v_1 = -1$, which is $\mathbf{v} = -\mathbf{i} + \mathbf{j}$.

(c) Find an orthogonal matrix P and a diagonal matrix D so that $A = P^t DP$.

Solution: We need normalised eigenvectors to make the rows of P and eigenvalues to make the diagonal entries of D. The eigenvectors we found both have length $\sqrt{2}$. So we divide them by $\sqrt{2}$ to make them unit vectors.

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} 7 & 0 \\ 0 & 5 \end{bmatrix}$$

7. The matrix $A = \begin{bmatrix} 2 & -1 \\ 3/2 & -1/2 \end{bmatrix}$ is diagonalisable and $A = SDS^{-1}$ where

$$S = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

(a) Calculate e^{Ax}.

Solution: We know $Ax = (SDS^{-1})x = S(Dx)S^{-1}$ (since x is a scalar) and Dx is the diagonal. In fact

$$Dx == \begin{bmatrix} x & 0 \\ 0 & \frac{x}{2} \end{bmatrix}.$$

Then

$$\begin{aligned} e^{Ax} &= Se^{Dx}S^{-1} \\ &= S \begin{bmatrix} e^x & 0 \\ 0 & e^{x/2} \end{bmatrix} S^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^x & 0 \\ 0 & e^{x/2} \end{bmatrix} S^{-1} \\ &= \begin{bmatrix} e^x & 2e^{x/2} \\ e^x & 3e^{x/2} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3e^x - 2e^{x/2} & -2e^x + 2e^{x/2} \\ 3e^x - 3e^{x/2} & -2e^x + 3e^{x/2} \end{bmatrix} \end{aligned}$$

(b) Find the general solution of the system of linear differential equations

$$\begin{cases} \dfrac{dy_1}{dx} = 2y_1 - y_2 \\ \dfrac{dy_2}{dx} = (3/2)y_1 - (1/2)y_2 \end{cases}$$

Solution: We know the general solution has the form

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \alpha_1 e^{\lambda_1 x}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 x}\mathbf{v}_2$$

where α_1, α_2 are arbitrary constants and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors for the eigenvalues λ_1 and λ_2 of A.

We know from $A = SDS^{-1}$ that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1/2$. We also know that the eigenvectors are the columns of S. So the general solution is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \alpha_1 e^x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 e^{x/2} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \alpha_1 e^x + 2\alpha_2 e^{x/2} \\ \alpha_1 e^x + 3\alpha_2 e^{x/2} \end{bmatrix}$$

We could write this $y_1(x) = \alpha_1 e^x + 2\alpha_2 e^{x/2}$, $y_2(x) = \alpha_1 e^x + 3\alpha_2 e^{x/2}$.

(c) Find the solution of the same system which satisfies the initial conditions $y_1(0) = 9$ and $y_2(0) = 6$.

Solution: We need to pick the constants α_1 and α_2 so that

$$\begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} \alpha_1 e^0 + 2\alpha_2 e^0 \\ \alpha_1 e^0 + 3\alpha_2 e^0 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ \alpha_1 + 3\alpha_2 \end{bmatrix}$$

So we solve

$$\begin{cases} \alpha_1 + 2\alpha_2 &= 9 \\ \alpha_1 + 3\alpha_2 &= 6 \end{cases}$$

and we can do that by row reducing

$$\begin{bmatrix} 1 & 2 & : 9 \\ 1 & 3 & : 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & : & 9 \\ 0 & 1 & : & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & : & 15 \\ 0 & 1 & : & -3 \end{bmatrix}$$

So $\alpha_1 = 15$ and $\alpha_2 = -3$.

That gives $y_1(x) = 15e^x - 6e^{x/2}$, $y_2(x) = 15e^x - 9e^{x/2}$.

8. (a) A loaded die has the following probabilities of showing the numbers 1–6 after a throw:

$$\frac{3}{17}, \frac{2}{17}, \frac{4}{17}, \frac{2}{17}, \frac{1}{17}, \frac{5}{17}$$

(in that order). Find the probability that a number ≥ 5 will show after the die is thrown.

A random variable X associated with the outcome has the values

$$X(1) = X(3) = 4, X(2) = X(6) = -2, X(4) = X(5) = 3.$$

Find the mean of the random variable.

Solution: The probability of a number ≥ 5 is

$$P(5) + P(6) = \frac{1}{17} + \frac{5}{17} = \frac{6}{17}$$

The mean of the random variable X is

$$\begin{aligned} & P(1)X(1) + P(2)X(2) + P(3)X(3) + P(4)X(4) + P(5)X(5) + P(6)X(6) \\ = \; & \frac{3}{17}4 + \frac{2}{17}(-2) + \frac{4}{17}4 + \frac{2}{17}3 + \frac{1}{17}3 + \frac{5}{17}(-2) \\ = \; & \frac{23}{17} \end{aligned}$$

(b) If the number of alpha particles detected per second by a particular detector obeys a Poisson distribution with mean $\mu = 0.8$, what is the probability that at most 2 particles are detected in a given second?

Solution: We know that the probability that the number detected is eactly n is

$$P(n) = \frac{\mu^n}{n!} e^{-\mu}$$

and what we want is

$$P(\{0, 1, 2\}) = P(0) + P(1) + P(2) = \frac{1}{0!} e^{-\mu} + \frac{\mu}{1!} e^{-\mu} + \frac{\mu^2}{2!} e^{-\mu}$$

Now $e^{-\mu} = e^{-0.8} = 0.449329$. So the result is

$$P(\{0, 1, 2\}) = \left(1 + \mu + \frac{\mu^1}{2}\right) e^{-\mu} = (1+0.8+0.32)(0.449329) = 0.952577$$

(c) A factory produces bottles of a juice that are sold as 0.33 litre bottles. A good model is that the quantity of juice in a bottle obeys a normal distribution with mean 0.34 (litres) and standard deviation 0.03. What proportion of the bottles have at least 0.32 litres in them?

Solution: We know the probality that the result of a normal distribution with mean μ and standard deviation σ is less than x can be related to the standard normal by

$$P(\text{result } < x) = F_{\mu,\sigma}(x) = F_{0,1}\left(\frac{x - \mu}{\sigma}\right)$$

In our case

$$
\begin{aligned}
P([0.32,\infty) &= 1 - P(\text{result } < 0.32) \\
&= 1 - F_{0.34,0.03}(0.32) \\
&= 1 - F_{0,1}\left(\frac{0.32-0.34}{0.03}\right) \\
&= 1 - F_{0,1}(-0.666) \\
&= F_{0,1}(0.666) \\
&= 0.747294
\end{aligned}
$$

So 74.73% have at least 0.32 litres in them.