Part 2 (1S2)

5. (a) Write an augmented matrix corresponding to the following system of linear equations:

$$\begin{array}{rcrcrcrcr} 7x_1 & - & 4x_2 & - & 2x_3 & & & = & -12 \\ 3x_1 & & & + & 10x_3 & - & 3x_4 & = & 8 \\ & & 2x_2 & + & 11x_3 & - & 6x_4 & = & -27 \\ x_1 & + & 2x_2 & - & x_3 & - & x_4 & = & -3 \end{array}$$

Solution:

$$\left[\begin{array}{rrrrcr} 7 & -4 & -2 & 0 & : & -12 \\ 3 & 0 & 10 & -3 & : & 8 \\ 0 & 2 & 11 & -6 & : & -27 \\ 1 & 2 & -1 & -1 & : & -3 \end{array}\right]$$

(b) Using the Gauss Jordan method strictly, find a reduced row-echelon form for the (augmented) matrix

$$\left[\begin{array}{rrrrrcr} 0 & -2 & 0 & 7 & 1 & : & 12 \\ 1 & -5 & 3 & 6 & 2 & : & 14 \\ 2 & -5 & 6 & -5 & -1 & : & -1 \end{array}\right]$$

Solution:

$$\left[\begin{array}{rrrrrcr} 1 & -5 & 3 & 6 & 2 & : & 14 \\ 0 & -2 & 0 & 7 & 1 & : & 12 \\ 2 & -5 & 6 & -5 & -1 & : & -1 \end{array}\right. \left|\begin{array}{l} \text{oldRow2} \\ \text{oldRow1} \\ \end{array}\right.$$

$$\left[\begin{array}{rrrrrcr} 1 & -5 & 3 & 6 & 2 & : & 14 \\ 0 & -2 & 0 & 7 & 1 & : & 12 \\ 0 & 5 & 0 & -17 & -5 & : & -29 \end{array}\right. \left|\begin{array}{l} \\ \\ \text{oldRow3} - 2 \times \text{oldRow1} \end{array}\right.$$

$$\left[\begin{array}{rrrrrcr} 1 & -5 & 3 & 6 & 2 & : & 14 \\ 0 & 1 & 0 & -\frac{7}{2} & -\frac{1}{2} & : & -6 \\ 0 & 5 & 0 & -17 & -5 & : & -29 \end{array}\right. \left|\begin{array}{l} \\ \text{oldRow2} \times \left(-\frac{1}{2}\right) \\ \end{array}\right.$$

$$\left[\begin{array}{rrrrrcr} 1 & -5 & 3 & 6 & 2 & : & 14 \\ 0 & 1 & 0 & -\frac{7}{2} & -\frac{1}{2} & : & -6 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{5}{2} & : & 1 \end{array}\right. \left|\begin{array}{l} \\ \\ \text{oldRow3} - 5 \times \text{oldRow2} \end{array}\right.$$

$$\left[\begin{array}{rrrrrcr} 1 & -5 & 3 & 6 & 2 & : & 14 \\ 0 & 1 & 0 & -\frac{7}{2} & -\frac{1}{2} & : & -6 \\ 0 & 0 & 0 & 1 & -5 & : & 2 \end{array}\right. \left|\begin{array}{l} \\ \\ \text{oldRow3} \times 2 \end{array}\right.$$

$$\left[\begin{array}{rrrrrcr} 1 & -5 & 3 & 0 & 32 & : & 2 \\ 0 & 1 & 0 & 0 & -18 & : & 1 \\ 0 & 0 & 0 & 1 & -5 & : & 2 \end{array}\right. \left|\begin{array}{l} \text{oldRow1} - 6 \times \text{oldRow3} \\ \text{oldRow2} + \frac{7}{2} \times \text{oldRow3} \\ \end{array}\right.$$

$$\left[\begin{array}{cccccc|l} 1 & 0 & 3 & 0 & -58 & :7 & \text{oldRow1} + 5 \times \text{oldRow2} \\ 0 & 1 & 0 & 0 & -18 & :1 & \\ 0 & 0 & 0 & 1 & -5 & :2 & \end{array}\right.$$

This is in reduced row-echelon form. (Gauss-Jordan completed.)

6. (a) For $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$, calculate the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$ and the projection $\mathrm{proj}_{\mathbf{w}}(\mathbf{v})$ of $\mathbf{v}$ along the direction of $\mathbf{w}$.

Solution: We know

$$\begin{aligned} \cos\theta &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \\ &= \frac{2(2) - 5(-3) + 3(-5)}{\sqrt{2^2 + (-5)^2 + 3^2 2^2 + (-3)^2 + (-5)^2}} \\ &= \frac{4}{\sqrt{38}\sqrt{38}} = \frac{4}{38} = \frac{2}{19} \end{aligned}$$

and

$$\begin{aligned} \mathrm{proj}_{\mathbf{w}}(\mathbf{v}) &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} \\ &= \frac{4}{38}\mathbf{w} \\ &= \frac{2}{19}(2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}) \\ &= \frac{4}{19}\mathbf{i} - \frac{6}{19}\mathbf{j} - \frac{10}{19}\mathbf{k} \end{aligned}$$

(b) Find both the parametric and cartesian equations for the line in space which passes through the both of the point $(4, 2, 3)$ and $(0, 1, 1)$.

Solution: If $P = (4, 2, 3)$ and $Q = (0, 1, 1)$ are the two points, and we take their position vectors $\mathbf{P} = 4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{Q} = \mathbf{j} + \mathbf{k}$, then $\vec{PQ} = \mathbf{Q} - \mathbf{P}$ is a vector parallel to the line. So a vector parallel to the line is

$$\mathbf{Q} - \mathbf{P} = (\mathbf{j} + \mathbf{k}) - (4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = -4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

So we can write down the parametric equations using the point P on the line and this vector parallel to the line as

$$\begin{aligned} x &= 4 - 4t \\ y &= 2 - t \\ z &= 3 - 2t. \end{aligned}$$

Solving each of these for t we get

$$\frac{x-4}{-4} = t, \quad \frac{y-2}{-1} = t, \quad \frac{z-3}{-2} = t$$

Thus the cartesian equations are

$$\frac{x-4}{-4} = \frac{y-2}{-1} = \frac{z-3}{-2}$$

(c) Find an equation for the plane in space that contains the point $(1, -1, 2)$ and is parallel to the plane

$$5x - 6y + z = 4.$$

Solution: A parallel plane has the same normal vector $5\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ and so has an equation

$$5x - 6y + z = c$$

for some c. Since $(1, -1, 2)$ must satisfy the equation, we have $5 + 6 + 2 = c$ or $c = 13$. So the answer is

$$5x - 6y + z = 13$$

7. (a) Let $\mathbf{x} = (3, 2, -3, 5, 3)$ and $\mathbf{y} = (2, 1, 0, -4, -2)$ (in $\mathbb{R}^5$). Compute $3\mathbf{x} - 4\mathbf{y}$ and the distance between $\mathbf{x}$ and $\mathbf{y}$.

Solution:

$$\begin{aligned} 3\mathbf{x} - 4\mathbf{y} &= 3(3, 2, -3, 5, 3) - 4(2, 1, 0, -4, -2) \\ &= (9, 6, -9, 15, 9) - (8, 4, 0, -16, -8) \\ &= (1, 2, -9, 31, 17) \end{aligned}$$

and

$$\begin{aligned} \text{distance}(\mathbf{x}, \mathbf{y}) &= \sqrt{(2-3)^2 + (1-2)^2 + (0+3)^2 + (-4-5)^2 + (-2-3)^2} \\ &= \sqrt{1 + 1 + 9 + 81 + 25} = \sqrt{117} \end{aligned}$$

(b) For

$$a = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 2 \\ 5 & 7 & 1 \end{bmatrix},$$

compute ab, ba and $ab - ba$.

Solution:

$$ab = \begin{bmatrix} 11 & 27 & 1 \\ -10 & -14 & -2 \\ 13 & 21 & 2 \end{bmatrix}$$

$$ba = \begin{bmatrix} -1 & -2 & 13 \\ -2 & -4 & 28 \\ 5 & -8 & 4 \end{bmatrix}$$

$$ab - ba = \begin{bmatrix} 12 & 29 & -12 \\ -8 & -10 & -30 \\ 8 & 29 & -2 \end{bmatrix}$$

8. (a) What output would be produced by the following `Mathematica` instruction, and what does it mean?

```
PrimeQ[36]
```

Solution: The output will be `False`, and the meaning is that the answer to the question "Is 36 a prime?" is No. (Of course $36 = 2 \times 18 = 4 \times 9 = 2^2 3^2$ is far from prime.)

(b) Write a `Mathematica` instruction to solve $3x^2 + 2x - a^2 = 0$ for x.

Solution:

```
Solve[ 3x^2 + 2x -a^2 == 0, x]
```

(c) Write a `Mathematica` instruction to graph $y = \frac{x^3+x-2}{x^2+1}$ for x in the range $-10 \leq x \leq 10$.

Solution:

```
Plot[(x^3 + x - 2)/(x^2 + 1), {x, -10, 10}]
```

(d) What do the following `Mathematica` instruction do?

```
y = x Cos[x] - x^2;
D[y, x]
```

Work out mathematically, in as much detail as you can, what the result will be (and give reasons).

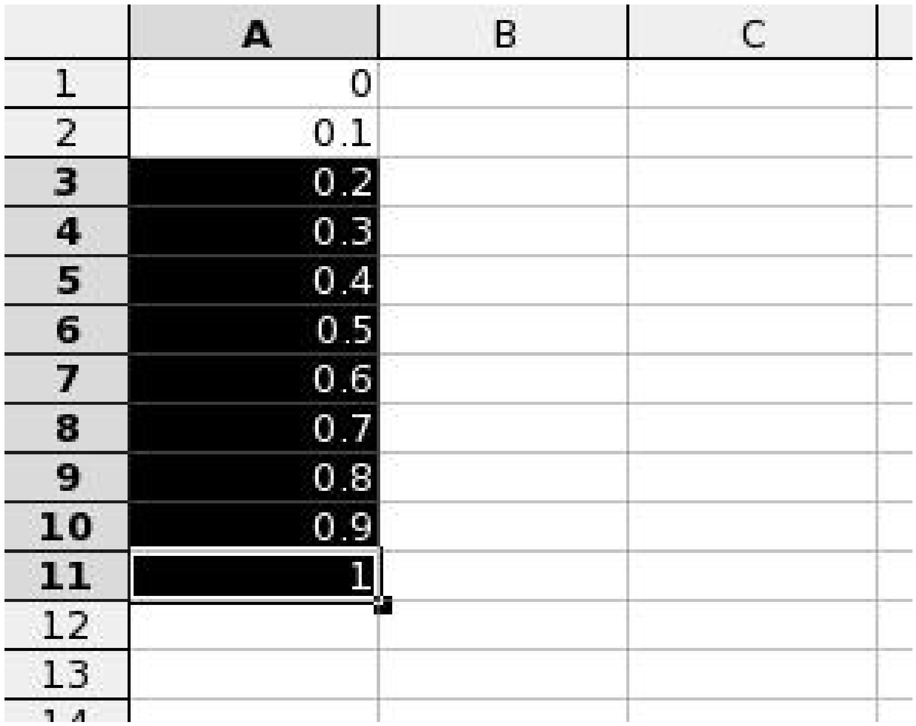
Solution: The first line is a definition of y as a shorthand for $x \cos x - x^2$ and the effect of the second line is to compute the derivative $\frac{dy}{dx}$. The result will be

$$\frac{dy}{dx} = \frac{d}{dx}(x \cos x - x^2) = \cos x - x \sin x - 2x$$

and Mathematica should report an answer

```
-2 x + Cos[x] - x Sin[x]
```

(e) The following shows a portion of a spreadsheet.



In order to compute in cells **B1** down to **B11** the cosines of the numbers in the cells to the left (that is the number $\cos 0$ in cell **B1**, $\cos 0.1$ in cell **B2**, etc), describe what you would do. Be specific in your description.

Having done that, you would you calculate the sum of the numbers in cells **B1** to **B11** in cell **B12** (with the use of a formula)? (That is, what formula would you use?)

Solution: You could enter in cell **B1** the formula `=cos(A1)`, then copy from cell **B1** and paste into cells **B2** to **B11**.

To get the sum in cell **B12**, enter in that cell the formula `=sum(B1:B11)`.