Chapter 8. Linear transformations

This material is a reduced treatment of what is in Anton & Rorres sections 4.2, 4.3, chapter 6 (mostly for the case of 3 dimensions) and chapter 7.

8.1 Introduction

Our aim first is to give a new way of looking at matrices. It is a development from ideas in section 5.7 where we expressed a system of linear equations as a single matrix equation

$$A\mathbf{x} = \mathbf{b}$$

Recall that a system of m linear equations in n unknowns

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

can be written as a single matrix equation $A\mathbf{x} = \mathbf{b}$ where A is an $m \times n$ matrix, $\mathbf{x}$ is an $n \times 1$ (column) matrix of unknowns and $\mathbf{b}$ is an $m \times 1$ column.

In single variable calculus we realise that it is convenient to use the language of functions to explain what happens when we are solving equations. The kind of examples we have there include

$$\begin{aligned} 2x + 3 &= 5 && \text{(linear)} \\ x^2 + 2x + 7 &= 2 && \text{(quadratic)} \\ x^3 + 4x^2 - 2x + 1 &= 4 && \text{(cubic)} \\ \cos x &= x && \text{(more complicated — can be} \\ &&& \text{written } \cos x - x = 0) \end{aligned}$$

We can think of all of these as $f(x) = 0$ or $f(x) = b$ for a suitable function $f(x)$.

Inspired by this experience we might be inclined to think of a system of linear equations

$$A\mathbf{x} = \mathbf{b}$$

in terms of functions.

The functions we need for this have to be functions of a vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or alternatively can be viewed as functions of n variables $(x_1, x_2, \ldots, x_n)$. In fact the language of functions is already designed to cope with these more complex situations.

While we become used to having functions $f(x)$ where the independent variable x is a number, and where $f(x)$ is usually given by a formula, this is not the definition of a function. We may be used to saying "let $f(x) = x^3 + 4x^2 - 2x + 1$ be a function" (so given by the formula) but this is not the only kind of function that is allowed.

In general a function $f\colon S \to T$ is something that has a *domain* set S and a *target* set T (more often called a *codomain* set T) and the definition says that a function f is a rule that assigns one and only one element $f(s) \in T$ to each $s \in S$. The rule does not have to be a formula, but it does have to be an unambiguous rule that works for every s in the domain set S. We usually refer to $f(s)$ as the 'value' of the function at s.

The most familiar examples are functions $f\colon \mathbb{R} \to \mathbb{R}$ where the domain is $S = \mathbb{R}$ (the whole real line) and the values are also real numbers. So we have functions like $f(x) = x\cos x$ or $f(x) = x^2 + 2x + 3$ where the rule is given by a formula. Sometimes we also encounter cases where the domain S is not all of $\mathbb{R}$, but only a part of $\mathbb{R}$. For instance the square root function $f(x) = \sqrt{x}$ gets in trouble if we try to take square roots of negative numbers. We would usually take the domain $S = [0, \infty) = \{x \in \mathbb{R} : x \geq 0\}$ and $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. (This might be a good place to remark that by the square root we mean the positive square root. So $\sqrt{4}$ means 2, not -2. If we did not do this the square root would not qualify as an ordinary function with values in $\mathbb{R}$. $f(x)$ to have just one value when we have a function.)

For function $f\colon \mathbb{R}^2 \to \mathbb{R}$, there are examples we have more or less seen already. We could have for example

$$f(\mathbf{x}) = \|\mathbf{x}\| = \text{ distance from } \mathbf{x} \text{ to } \mathbf{0}$$

or we could write the same example

$$f(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$$

when we take $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$ (or equivalently think of points $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$).

Another perfectly good example would be

$$f(x_1, x_2) = x_1^3 + x_2^4 - x_1$$

(which has no obvious geometrical interpretation, but makes sense as a rule that associates a value $f(x_1, x_2)$ to a point $(x_1, x_2) \in \mathbb{R}^2$).

An example that is a bit closer to the kinds of functions that will really interest us is

$$f(x_1, x_2) = 2x_1 - 3x_2$$

We can rewrite this using vectors as

$$f(x_1, x_2) = (2\mathbf{i} - 3\mathbf{j}) \cdot (x_1\mathbf{i} + x_2\mathbf{j}).$$

We'll be thinking of this example, or things very like it, in matrix language. The matrix product

$$\begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is a 1×1 matrix

$$[2x_1 - 3x_2]$$

and so has just one entry, the number $2x_1 - 3x_2$. If we are prepared to ignore the (arguably rather technical) difference between the number $2x_1 - 3x_2$ as a pure number and the 1×1 matrix $[2x_1 - 3x_2]$, then we can write the same example using matrix multiplication

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now that we've shown some examples in $\mathbb{R}^2$, we could show some examples in $\mathbb{R}^3$ without too much extra imagination. We could even look at functions $f\colon \mathbb{R}^n \to \mathbb{R}$ given by rules (or formulae) like

$$f(\mathbf{x}) = \|\mathbf{x}\|$$

which now looks like

$$f(x_1, x_2, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Another example, generalising the one we just gave with $2\mathbf{i} - 3\mathbf{j}$, would be to fix $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and define $f\colon \mathbb{R}^n \to \mathbb{R}$ by the rule

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$$

We could think of this example in matrix terms also as

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

if we are still prepared to ignore the distinction between a scalar and a 1×1 matrix.

When A is an $m \times n$ matrix the formula

$$f(\mathbf{x}) = A\mathbf{x}$$

applied to an $n \times 1$ (column)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

will produce an $m \times 1$ result. So the result can be interpreted as a point (or vector) in $\mathbb{R}^m$ (if we identify n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with $n \times 1$ column matrices, and similarly identify $m \times 1$ column matrices with points in $\mathbb{R}^m$). Our formula $f(\mathbf{x}) = A\mathbf{x}$ will then make sense to define a function

$$f\colon \mathbb{R}^n \to \mathbb{R}^m$$

Often people prefer to call these *transformation* rather than functions. Technically they are the same as functions (rules that work on some domain and produce values somewhere) but the idea of using the word 'transformation' instead of plain 'function' is partly due to the fact that the values are vectors rather than scalar values. There is also a geometrical way of thinking about what these transformations do, or look like, that we will try to explain in some examples soon. This viewpoint partly explains the 'transformation' word.

For now we will summarise what we have with a definition.

8.1.1 Definition. A *linear transformation* $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a function given by a rule

$$f(\mathbf{x}) = f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}$$

for some give $m \times n$ matrix A.

8.2 Examples of linear transformations

In the first examples we take $n = m = 2$, so that we have 2×2 matrices A and linear transformations $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the plane.

(i) $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$,

$$A\mathbf{x} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

So we have $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $f(\mathbf{x}) = 3\mathbf{x} =$ the vector 3 times as long as $\mathbf{x}$ in the same direction. Or, if we think in terms of points in the plane $f(\mathbf{x})$ is the point 3 times further from the origin than $\mathbf{x}$, in the same direction.

Thinking of what happens to the points of $\mathbb{R}^2$ (picture the plane) after you apply the transformation f to them, you see everything expands by a factor 3 away from the central point at the origin. This is sometimes called 'dilation' by a factor 3 (like looking through a magnifying glass).

(ii) $A = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$,

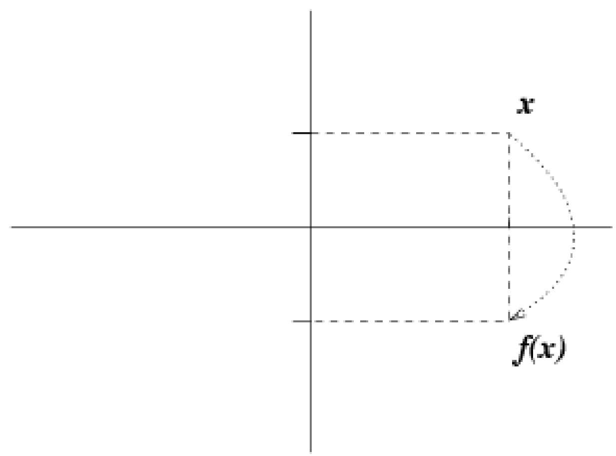
$$A\mathbf{x} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ x_2 \end{bmatrix}$$

The picture here is that the plane is expanded or stretched horizontally (in the x_1- or x-direction) by a factor 5, while the vertical scale is not affected. The vertical axis does not move.

(iii) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

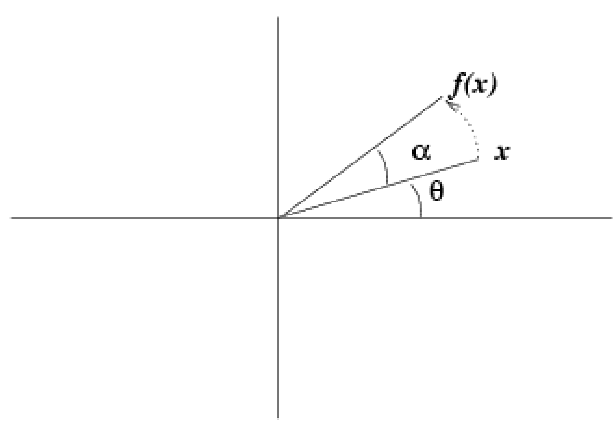
$$A\mathbf{x} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

The picture for this involves a mirror. The relation between $\mathbf{x}$ and $f(\mathbf{x})$ is that the point $f(\mathbf{x})$ arises by reflecting the original point $\mathbf{x}$ in the horizontal axis.



(iv) $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$,

$$A\mathbf{x} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\cos\alpha)x_1 - (\sin\alpha)x_2 \\ (\sin\alpha)x_1 + (\cos\alpha)x_2 \end{bmatrix}$$



In the picture the effect of this transformation is shown as a rotation by an angle α (radians) anticlockwise. To see why this is the case, it is easiest if we use polar coordinates. This means describing the position of a point $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ via two other numbers (t, θ) where $r = \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2} = \text{distance}(\mathbf{x}, \mathbf{0})$ and θ is the angle (in radians) from the positive x_1-axis (horizontal of x-axis) around anticlockwise to the radius from the origin to $\mathbf{x}$. In formulae, the relation is

$$(x_1, x_2) = (r\cos\theta, r\sin\theta)$$

If we apply the map, we get

$$\begin{aligned} f(\mathbf{x}) &= A\mathbf{x} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} (\cos\alpha)(r\cos\theta) - (\sin\alpha)(r\sin\theta) \\ (\sin\alpha)(r\cos\theta) + (\cos\alpha)(r\sin\theta) \end{bmatrix} \\ &= \begin{bmatrix} r(\cos\alpha\cos\theta - \sin\alpha\sin\theta) \\ r(\sin\alpha\cos\theta + \cos\alpha\sin\theta) \end{bmatrix} \\ &= \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix} \end{aligned}$$

At the last step above we used two trigonometric identities you may recall:

$$\begin{aligned} \cos(\alpha+\theta) &= \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ \sin(\alpha+\theta) &= \sin\alpha\cos\theta + \cos\alpha\sin\theta \end{aligned}$$

The result of the above calculation is that the polar coordinates for the image point $f(\mathbf{x})$ have the same r as the original point $\mathbf{x}$, so same distance from the origin, but the polar coordinates angle for $f(\mathbf{x})$ is $\theta + \alpha$, or an increase by α on the angle θ for $\mathbf{x}$. This justifies the picture.

8.3 Orthogonal matrices

Before we look at more examples, we point out that the last two examples, the reflection and the rotation, both have the property that

$$A^{-1} = A^t$$

In fact for the reflection we have

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = A^{-1} = A^t$$

and for the rotation, you can check out that

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \text{ has } A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A^t$$

So in both cases $AA^t = I_2 = A^tA$.

8.3.1 Definition. An $n \times n$ matrix A is called an *orthogonal matrix* if $AA^t = I_n = A^tA$ (which means that A is invertible and $A^{-1} = A^t$.

If we write n-tuples as columns

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix},$$

then the dot product

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

can be expressed in matrix terms if we are prepared to overlook the distinction between a scalar and a 1×1 matrix.

$$(\mathbf{v}^t)\mathbf{w} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = [v_1 w_1 + v_2 w_2 + \cdots + v_n w_n] = \mathbf{v} \cdot \mathbf{w}$$

Using this way of looking at things we can show that orthogonal matrices have some nice properties. (So these properties apply in particular to our 2×2 reflections and rotations.)

8.3.2 Proposition. *Let A be an $n \times n$ orthogonal matrix and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ the associated linear transformation $f(\mathbf{x}) = A\mathbf{x}$. Then, for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ we have*

(i) $(A\mathbf{v}) \cdot (A\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$

(ii) $f(\mathbf{v}) \cdot f(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ *(so that the orthogonal matrices preserve inner products, or the transformation associated with them preserves inner products)*

(iii) $\|A\mathbf{v}\| = \|\mathbf{v}\|$

(iv) $\|f(\mathbf{v})\| = \|\mathbf{v}\|$ *(so preserve distance from the origin)*

(v) $\|A\mathbf{v} - A\mathbf{w}\| = \|\mathbf{v} - \mathbf{w}\|$

(vi) $\|f(\mathbf{v}) - f(\mathbf{w})\| = \|\mathbf{v} - \mathbf{w}\|$ *(so preserve distances between points)*

(vii) *if θ denotes the angle between $\mathbf{v}$ and $\mathbf{w}$, then the angle between $f(\mathbf{v}) = A\mathbf{v}$ and $f(\mathbf{w}) = A\mathbf{w}$ is also θ (so preserves angles).*

Proof. Since we have mostly done the work already we can quickly show that all these properties are indeed correct.

(i) $(A\mathbf{v}) \cdot (A\mathbf{w}) = ((A\mathbf{v})^t)A\mathbf{w} = \mathbf{v}^t A^t A\mathbf{w} = \mathbf{v}^t I_n \mathbf{w} = (\mathbf{v}^t)\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$

(We used here the rule that the transpose of a product is the product of the transposes in the reverse order.)

(ii) $f(\mathbf{v}) \cdot f(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ is just the same thing again in a different notation.

(iii) $\|A\mathbf{v}\|^2 = (A\mathbf{v}) \cdot (A\mathbf{v}) = \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ (by the first part with $\mathbf{v} = \mathbf{w}$). So $\|A\mathbf{v}\| = \|\mathbf{v}\|$.

(iv) $\|f(\mathbf{v})\| = \|\mathbf{v}\|$ is just the same statement again.

(v) $\|A\mathbf{v} - A\mathbf{w}\| = \|A(\mathbf{v} - \mathbf{w}\| = \|\mathbf{v} - \mathbf{w}\|$ (using properties of matrix multiplication and also the last statement).

(vi) $\|f(\mathbf{v}) - f(\mathbf{w})\| = \|\mathbf{v} - \mathbf{w}\|$ is just the same thing again in different notation.

(vii) The angle θ between $\mathbf{v}$ and $\mathbf{w}$ comes about (by definition when $n > 3$ or because we proved it for $n = 2$ and $n = 3$) from

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta,$$

But, using what we have just shown we can deduce

$$(A\mathbf{v}) \cdot (A\mathbf{w}) = \mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta = \|A\mathbf{v}\|\|A\mathbf{w}\| \cos\theta,$$

which says that the angle between $f(\mathbf{v}) = A\mathbf{v}$ and $f(\mathbf{w}) = A\mathbf{w}$ is also θ.

□

8.3.3 Proposition. *If A is an orthogonal matrix, the* $\det(A) = \pm 1$.

Proof. (See tutorial sheet 14, q 4).

If A is orthogonal, $AA^t = I_n$. So $\det(AA^t) = \det(I_n) = 1$. So

$$\det(A)\det(A^t) = 1.$$

But $\det(A^t) = \det(A)$. Thus we get

$$(\det(A))^2 = 1$$

and that tells us $\det(A)$ has to be either 1 or -1. □

8.3.4 Proposition. *The 2×2 orthogonal matrices of determinant 1 are exactly the rotation matrices.*

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \qquad (\alpha \in \mathbb{R},\ \text{or } 0 \leq \alpha < 2\pi)$$

The 2×2 orthogonal matrices of determinant -1 are exactly the products

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

of a rotation and a reflection matrix.

Proof. We can see quite easily that the rotation matrices do have determinant 1, and they are all orthogonal as we already saw.

$$\det \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = (\cos\alpha)^2 - (-\sin\alpha)(\sin\alpha) = \cos^2\alpha + \sin^2\alpha = 1$$

Now if we start with some orthogonal 2×2 matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

we want to show it is a rotation matrix, or a rotation matrix times a reflection.

Notice that

$$B\mathbf{i} = B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

is the first column of B. So it must be a unit vector (or the point (b_{11}, b_{21}) must be on the unit circle) since we know $\|B\mathbf{i}\| = \|\mathbf{i}\| = 1$ (because of Proposition 8.3.2 (iii)). So if we write this point in polar coordinates we can say

$$\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$$

for some β.

Next notice that

$$B\mathbf{j} = B \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix}$$

is the second column of B. It must be also be a unit vector and it has to be perpendicular to $B\mathbf{i} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$ (since $\mathbf{i} \perp \mathbf{j}$ and orthogonal matrices preserve angles — Proposition 8.3.2 (vii)).

But there are just two unit vectors in the plane perpendicular to $\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$. You can see that from a picture, and you can also check that those two vectors are

$$\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \text{ and } -\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} = \begin{bmatrix} \sin\beta \\ -\cos\beta \end{bmatrix}$$

So these are the two choices for the second column of B and we have

$$B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \text{ or } B = \begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix}$$

In the first case we have a rotation matrix (and $\det(B) = 1$) while the second possibility can be written

$$B = \begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

In this second case we have

$$\det(B) = \det\left(\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\right)\det\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right) = (1)(-1) = -1$$

So the sign of the determinant distinguished between the two cases. □

8.3.5 Remark. We can think of the transformation in the second case (when $\det(B) = -1$ above) as the result of first applying a reflection in the horizontal axis and then doing a rotation. So the transformation in this case is a reflection followed by a rotation.

We won't justify it, but in fact this combination of a reflection and a rotation is just a reflection in some line through the origin. (The one inclined at angle $\beta/2$ in fact.)

8.4 Rotations in space

Can we say anything about rotations in space? We described rotations in the plane $\mathbb{R}^2$ about the origin. What can we say in $\mathbb{R}^3$?

We need an axis to rotate around in space. If we choose a convenient axis, then it is quite easy to see what to do. For example, if we rotate around the z-axis, the z-coordinates (or altitudes above the horizontal x-y plane) will stay constant and the rotation will affect the x and y coordinates in the same way as a rotation of the plane about the origin. We can then use what we had above about rotations in $\mathbb{R}^2$ and show that the matrix

$$\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

corresponds to rotation about the z-axis by the angle α.

To see this more clearly look at

$$\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}(\cos\alpha)x - (\sin\alpha)y\\ (\sin\alpha)x + (\cos\alpha)y\\ z\end{bmatrix}$$

We see that (x, y) is rotated by α anticlockwise (from the x-axis towards the y-axis) to give

$$((\cos\alpha)x - (\sin\alpha)y, (\sin\alpha)x + (\cos\alpha)y)$$

and the height z stays fixed.

Here we used the fact that the axis was one of the coordinate axes. We can manage in a similar way if the axis is one of the other axes. For example rotation about the x-axis by an angle α (from y towards the z-axis) is given by the matrix

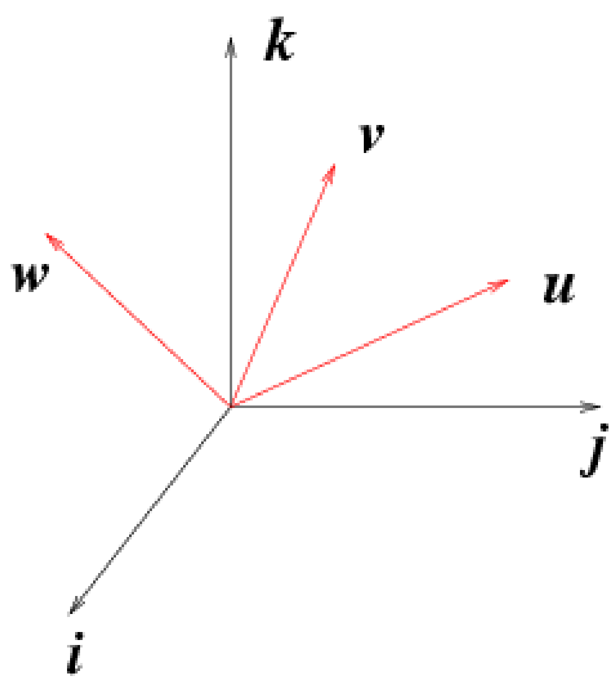
$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix}$$

But what about another axis? We would need a way to describe the direction of the axis and giving a nonzero vector $\mathbf{u}$ parallel to the axis seems like a handy way to describe that. Since the length of the vector does not seem to be useful, we can assume $\mathbf{u}$ is a unit vector. (If it is not, divide it by its length $\|\mathbf{u}\|$, which means replace the original $\mathbf{u}$ by $\mathbf{u}/\|\mathbf{u}\|$.)

One 'solution' is to choose 3 axes so that $\mathbf{u}$ is a unit vector in the direction of one of the axes. Then we have just seen how to write down a matrix to implement the rotation.

While this is an attractive solution, it is not really a complete solution. While in principle it is open to us to choose our coordinate axes to suit our problem (sometimes people call it choosing a frame of reference), mostly we are given the frame of reference earlier and we want to compute in terms of the given frame (or the given coordinates). What we really need is a way to go back and forth from the convenient coordinates to the original ones.

Suppose then we are given 3 perpendicular axes with the same origin as the original 3 axes. We will not discuss axes or frames with a different origin than the original. So, suppose we are thinking about describing the effect of a rotation about an axis through the origin in the direction of a unit vector $\mathbf{u}$. We assume we have 2 other axes perpendicular to each other and to $\mathbf{u}$. We can worry later about how to calculate them, but it is not hard to visualise $\mathbf{u}$ along with 2 other directions $\mathbf{v}$ and $\mathbf{w}$ so that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are all perpendicular unit vectors.



To make it more clear, imagine $\mathbf{u}$ is already known. Graphically, you can think of the plane through the origin perpendicular to $\mathbf{u}$ and visualise two unit vectors $\mathbf{v}$ and $\mathbf{w}$ in that plane and perpendicular to each other. We'll explain later how we might calculate $\mathbf{v}$ and $\mathbf{w}$ using components, starting from $\mathbf{u}$ given in components. (For the moment we will not worry about this aspect.)

Now that we have 3 perpendicular unit vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, we will write them in components

$$\begin{aligned} \mathbf{u} &= u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \\ \mathbf{v} &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}, \\ \mathbf{w} &= w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}. \end{aligned}$$

We claim that the matrix P made by using these vectors as the rows of P is an orthogonal matrix. That is

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

is orthogonal. The reason is that when we compute

$$PP^t = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

we multiply rows of P into columns of P^t and that means we end up taking dot products between rows of P. We get

$$PP^t = \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} & \mathbf{u}\cdot\mathbf{w} \\ \mathbf{v}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{w} \\ \mathbf{w}\cdot\mathbf{u} & \mathbf{w}\cdot\mathbf{v} & \mathbf{w}\cdot\mathbf{w} \end{bmatrix} = \begin{bmatrix} \|\mathbf{u}\|^2 & 0 & 0 \\ 0 & \|\mathbf{v}\|^2 & 0 \\ 0 & 0 & \|\mathbf{w}\|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

because the 3 vectors are perpendicular to one another and are unit vectors (length 1).

Since P is a square matrix and $PP^t = I_3$, then $P^tP = I_3$ is automatically true also and so P^t is P^{-1}.

We will show soon how to make use of the observation that P is orthogonal. One thing we can see is that

$$P\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} \\ \mathbf{v}\cdot\mathbf{u} \\ \mathbf{w}\cdot\mathbf{u} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{i}$$

and similarly

$$P\mathbf{v} = \mathbf{j}, \quad P\mathbf{w} = \mathbf{k}.$$

If we multiply the 3 equations

$$P\mathbf{u} = \mathbf{i}, \quad P\mathbf{v} = \mathbf{j} \text{ and } P\mathbf{w} = \mathbf{k} \tag{8.4.1}$$

by $P^{-1} = P^t$ on the left we get

$$\mathbf{u} = P^t\mathbf{j}, \quad \mathbf{v} = P^t\mathbf{j} \text{ and } \mathbf{w} = P^t\mathbf{k}. \tag{8.4.2}$$

In words (8.4.1) says that multiplication by P sends the new basis vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ back to the standard ones $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, while (8.4.2) says that multiplication by P^t does the opposite.

What we need though is to relate coordinates of any $\mathbf{x} \in \mathbb{R}^3$ with respect to the new $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ axes to coordinates of $\mathbf{x}$ is the standard axes.

Notice first that it is possible to express any vector $\mathbf{x}$ is $\mathbb{R}^3$ as a combination

$$\mathbf{x} = \xi_1\mathbf{u} + \xi_2\mathbf{v} + \xi_3\mathbf{w}$$

with components ξ_1, ξ_2, ξ_3 (scalars). Look back at section 3.6 and 3.6 when we showed that every vector $\mathbf{x}$ must be able to be expressed as

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

for some $x_1, x_2, x_3 \in \mathbb{R}$. We argued then in a geometrical way, based on the given axes. If we argue now in the same way, using the new $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ axes, we find that there must be scalars ξ_1, ξ_2, ξ_3.

That is a geometrical argument, which says something can be done. But in practice we need a way to calculate ξ_1, ξ_2, ξ_3. There is a nice trick for this using dot products. Remember, we know there are numbers ξ_1, ξ_2, ξ_3 but we want a way to calculate them. Starting from

$$\mathbf{x} = \xi_1\mathbf{u} + \xi_2\mathbf{v} + \xi_3\mathbf{w}$$

we can compute

$$\begin{aligned} \mathbf{x}\cdot\mathbf{u} &= (\xi_1\mathbf{u} + \xi_2\mathbf{v} + \xi_3\mathbf{w})\cdot\mathbf{u} \\ &= \xi_1\mathbf{u}\cdot\mathbf{u} + \xi_2\mathbf{v}\cdot\mathbf{u} + \xi_3\mathbf{w}\cdot\mathbf{u} \\ &= \xi_1 + 0 + 0 \\ &= \xi_1 \end{aligned}$$

and similarly we can check

$$\mathbf{x}\cdot\mathbf{v} = \xi_2 \text{ and } \mathbf{x}\cdot\mathbf{w} = \xi_3.$$

In matrix terms, we have

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}\cdot\mathbf{u} \\ \mathbf{x}\cdot\mathbf{v} \\ \mathbf{x}\cdot\mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{u}\cdot\mathbf{x} \\ \mathbf{v}\cdot\mathbf{x} \\ \mathbf{w}\cdot\mathbf{x} \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We can thus add to our earlier observations about P and P^t the following.

8.4.1 Proposition. *If* $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = \xi_1\mathbf{u} + \xi_2\mathbf{v} + \xi_3\mathbf{w}$, *where* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *are three mutually perpendicular unit vectors, and if*

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

then

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{8.4.3}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = P^t \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \tag{8.4.4}$$

The second statement (8.4.4) follows by multiplying both sides of (8.4.3) by $P^t = P^{-1}$.

8.4.2 Remark. Note that (8.4.2) said multiplication by P^t sends the original basis vectors **i**, **j** and $\mathbf{k}$ to the 'new' basis vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. But (8.4.3) says that P maps coordinates in the standard basis to coordinates in the new basis.

We could refer to P as the change of basis matrix, but there is also a case for giving that name to P^t (which has $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ as its columns).

8.4.3 Rotations in 3 dimensions. We can now write down a rotation around an axis in the direction $\mathbf{u}$ by an angle θ (a rotation by θ from $\mathbf{v}$ towards $\mathbf{w}$ in fact). In the coordinates of the new axes, we can use what we did earlier. Rotation sends the point with coordinates $\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$ to the point with coordinates

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

Let us write

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \tag{8.4.5}$$

for the coordinates of the rotated point in the $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ axes. Let us also name this rotated point $\mathbf{y}$.

We are saying then that when we apply the rotation to

$$\mathbf{x} = \xi_1\mathbf{u} + \xi_2\mathbf{v} + \xi_3\mathbf{w}$$

the result is

$$\mathbf{y} = \eta_1\mathbf{u} + \eta_2\mathbf{v} + \eta_3\mathbf{w}.$$

What we want is to be able to do all this in terms of coordinates with respect to the original basis **i**, **j** and **k**. We can figure this out using what we did before about the matrix P.

If $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ in standard coordinates, we know from (8.4.3)

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

On the other hand if $\mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$ is the rotated point expressed in standard coordinates, then (8.4.4) tells us

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = P^t \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

If we put these last two equations together with (8.4.5) we get

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = P^t \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The summary then is this

8.4.4 Proposition. *The matrix for a rotation around the axis through the origin in the direction* $\mathbf{u}$*, has the form*

$$R = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} P \tag{8.4.6}$$

where $\mathbf{u}$ *is assumed to be a unit vector,* $\mathbf{v}$ *and* $\mathbf{w}$ *are two other unit vectors perpendicular to each other and to* $\mathbf{u}$*, and the angle* θ *is measured from* $\mathbf{v}$ *towards* $\mathbf{w}$*. Here*

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$*,* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ *and* $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$*.*

8.4.5 Remarks. If we apply the rotation R from (8.4.6) to $\mathbf{u}$ we get

$$\begin{aligned} R\mathbf{u} &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} P\mathbf{u} \\ &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \mathbf{i} \\ & \qquad \text{(using } P\mathbf{u} = \mathbf{i} \text{ from (8.4.1)} \\ &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= P^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= P^t\mathbf{i} \\ &= \mathbf{u} \end{aligned}$$

(using $P^t\mathbf{i} = \mathbf{u}$ from (8.4.2) at the last step).

So $\mathbf{u}$ is fixed by R. But that is right because $\mathbf{u}$ is the axis.

Let's look now at $R\mathbf{v}$. Following steps that are similar till near the end, we get

$$\begin{aligned}
R\mathbf{v} &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} P\mathbf{v} \\
&= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \mathbf{j} \\
&= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
&= P^t \begin{bmatrix} 0 \\ \cos\theta \\ -\sin\theta \end{bmatrix} \\
&= P^t((\cos\theta)\mathbf{j} - (\sin\theta)\mathbf{k}) \\
&= (\cos\theta)P^t\mathbf{j} - (\sin\theta)P^t\mathbf{k} \\
&= (\cos\theta)\mathbf{v} - (\sin\theta)\mathbf{w}
\end{aligned}$$

Again, this is what we should expect to get when we think in terms of the axes $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

We'll give some terminology now. It is standard terminology and it deals with concepts that arose above.

8.4.6 Definition. Vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ are called *orthogonal* if they are all perpendicular to one another. That is if $\mathbf{u} \perp \mathbf{v}$, $\mathbf{u} \perp \mathbf{w}$ and $\mathbf{v} \perp \mathbf{w}$. (Using dot products we can express this $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$.)

Vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ are called *orthonormal* if they are orthogonal and they are also all unit vectors. (So $\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{w} = 1$ as well as orthogonality.)

This terminology (orthogonal and also orthonormal) is used for different numbers of vectors, not just for 3 vectors. However there is not room for more than 3 orthonormal vectors in $\mathbb{R}^3$ and room for only 2 in $\mathbb{R}^2$. So if we are in space, we can only have 1, 2 or 3 orthonormal vectors. In $\mathbb{R}^n$ we can have more.

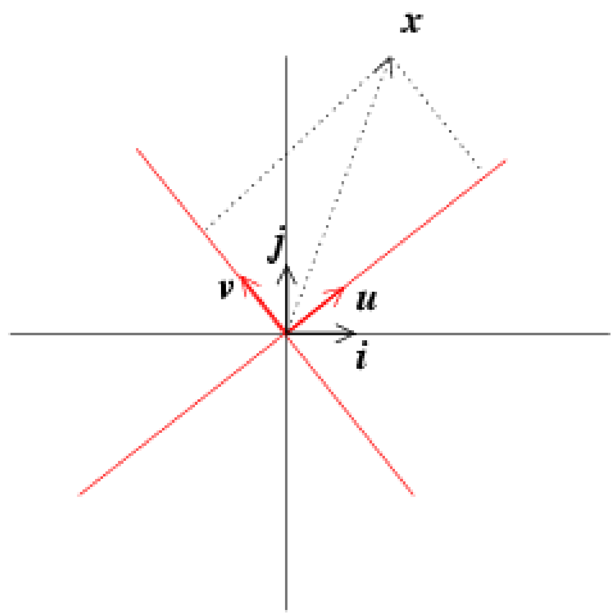
The definition of orthogonality is that each vector listed should be perpendicular to every other one. For orthonormal, each one should be a unit vector in addition.

8.4.7 Remark. We saw above that if $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are 3 orthonormal vectors in $\mathbb{R}^3$ then we can write every $\mathbf{x} \in \mathbb{R}^3$ as a combination

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u})\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\mathbf{v} + (\mathbf{x} \cdot \mathbf{w})\mathbf{w}$$

In two dimensions we have a similar fact. If $\mathbf{u}$ and $\mathbf{v}$ are orthonormal vectors in $\mathbb{R}^2$ then we can write every $\mathbf{x} \in \mathbb{R}^2$ as a combination

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u})\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\mathbf{v}$$



In this 2-dimensional setting, if we write $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ then the matrix P made by using the coordinates of $\mathbf{u}$ and $\mathbf{v}$ as its rows

$$P = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

has properties like we had in the 3×3 case earlier. These are

(i) P is an orthogonal matrix

(ii) $P\mathbf{u} = \mathbf{i}$ and $P\mathbf{v} = \mathbf{j}$.

(iii) $P^t\mathbf{i} = \mathbf{u}$ and $P^t\mathbf{j} = \mathbf{v}$.

(iv) For $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,

$$P\mathbf{x} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x} \cdot \mathbf{u} \\ \mathbf{x} \cdot \mathbf{v} \end{bmatrix}$$

gives the components of $\mathbf{x}$ in the $\mathbf{u}$-$\mathbf{v}$ basis (or frame).

Our next goal is to be able to complete the programme we started of writing down the matrix R from (8.4.6) starting with the axis $\mathbf{u}$ and the angle θ. What we did in (8.4.6) is a solution to that but it requires us to know the other two vectors $\mathbf{v}$ and $\mathbf{w}$ (so that the 3 together make up an orthonormal triple of vectors). As explained before, it is not hard to visualise $\mathbf{v}$ and $\mathbf{w}$ but we need a way to calculate them if we want to write down R.

The Gram-Schmidt method, which we now explain, is a useful method for other things. It starts with a number of vectors and 'straightens them out' so as to make them orthonormal to one another. In our case we will have 3 vectors.

In order to use it we need to have 3 vectors to start with, let's call them $\mathbf{u}$, $\mathbf{r}$ and $\mathbf{s}$. The vector $\mathbf{u}$ will be the one we know, the direction of the axis of rotation. We need $\mathbf{r}$ and $\mathbf{s}$ so that none of the 3 vectors lies in the same plane as the other two. This might sound hard, but it will work to take $\mathbf{r} = \mathbf{i}$ and $\mathbf{s} = \mathbf{j}$ unless $\mathbf{u}$ is a combination of $\mathbf{i}$ and $\mathbf{j}$ (with no $\mathbf{k}$ component).

If the axis $\mathbf{u} = (1/\sqrt{5})(\mathbf{i} + 2\mathbf{j})$ for example, we can take $\mathbf{r} = \mathbf{i}$ and $\mathbf{s} = \mathbf{k}$. So finding $\mathbf{r}$ and $\mathbf{s}$ is not a big deal.

8.4.8 Gram-Schmidt procedure. Starting with 3 vectors $\mathbf{u}$, $\mathbf{r}$ and $\mathbf{s}$, the procedure makes 3 orthonormal vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. It is a procedure with 3 steps (or really 2).

Step 1: If $\mathbf{u}$ is not a unit vector already, replace it by $(1/\|\mathbf{u}\|)\mathbf{u}$ (the unit vector with the same direction).

Step 2: Take

$$\mathbf{v} = \frac{\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}}{\|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}\|}$$

Step 3: Take

$$\mathbf{w} = \frac{\mathbf{s} - (\mathbf{s} \cdot \mathbf{u})\mathbf{u} - (\mathbf{s} \cdot \mathbf{v})\mathbf{v}}{\|\mathbf{s} - (\mathbf{s} \cdot \mathbf{u})\mathbf{u} - (\mathbf{s} \cdot \mathbf{v})\mathbf{v}\|}$$

The process is straightforward enough, though the calculations can take a bit of time to do. Here is an explanation of what is going on.

Step 1 is clear enough, perhaps. In our situation we will probably have a unit vector $\mathbf{u}$ anyhow and so step 1 will not be needed. But the rest of the steps need $\mathbf{u}$ to be a unit vector.

The numerator $\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}$ in step 2 works out as the difference

$$\mathbf{r} - \mathrm{proj}_{\mathbf{u}}(\mathbf{r})$$

and so is perpendicular to $\mathbf{u}$. (Look back at the pictures in section 3.11 to see why it is perpendicular.) Without thinking in terms of projections we can calculate

$$(\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}) \cdot \mathbf{u} = \mathbf{r} \cdot \mathbf{u} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u} \cdot \mathbf{u} = \mathbf{r} \cdot \mathbf{u} - (\mathbf{r} \cdot \mathbf{u})1 = 0$$

to see that they are indeed perpendicular. The numerator $\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}$ can't be zero because of the assumption that $\mathbf{r}$ is not in the same plane as $\mathbf{u}$ and $\mathbf{s}$. So, when we divide the numerator vector by its length we get a unit vector $\mathbf{v}$ which is perpendicular to $\mathbf{u}$.

An important little fact is that the plane of $\mathbf{u}$ and $\mathbf{v}$ is the same as the plane of $\mathbf{u}$ and $\mathbf{s}$, so that $\mathbf{r}$ is not in that plane.

At step 3, the numerator vector turns out to be perpendicular to both $\mathbf{u}$ and to $\mathbf{v}$. That can be checked using dot products. For example

$$\begin{aligned}\big(\mathbf{s} - (\mathbf{s} \cdot \mathbf{u})\mathbf{u} - (\mathbf{s} \cdot \mathbf{v})\mathbf{v}\big) \cdot \mathbf{u} &= \mathbf{s} \cdot \mathbf{u} - (\mathbf{s} \cdot \mathbf{u})\mathbf{u} \cdot \mathbf{u} - (\mathbf{s} \cdot \mathbf{v})\mathbf{v} \cdot \mathbf{u} \\ &= \mathbf{s} \cdot \mathbf{u} - (\mathbf{s} \cdot \mathbf{u})1 - 0 \\ &= 0\end{aligned}$$

Similarly the dot product with $\mathbf{v}$ works out as 0.

Dividing by the length gives a unit vector. We will not be dividing by zero since that would mean that $\mathbf{s}$ would be the same as $(\mathbf{s}\cdot\mathbf{u})\mathbf{u} + (\mathbf{s}\cdot\mathbf{v})\mathbf{v}$ and that would say $\mathbf{s}$ would be in the plane of $\mathbf{u}$ and $\mathbf{v}$ (which it is not).

8.4.9 Right handed and left handed. So now $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are orthonormal vectors and one of them is $\mathbf{u}$. So

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

is an orthogonal matrix.

But there is another (apparently equally good) triple of vectors $\mathbf{u}$, $\mathbf{w}$ and $\mathbf{v}$ and another orthonormal change of basis matrix

$$Q = \begin{bmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

where the order of $\mathbf{v}$ and $\mathbf{w}$ is reversed.

Which is better? Or is there any difference?

We know from tutorial sheet 14 (q 4) that orthogonal matrices have determinant ± 1. [The argument applied to P is that $PP^t = I_3$ implies $\det(PP^t) = \det(I_3) = 1$. But $\det(PP^t) = \det(P)\det(P^t)$ by the rule about determinants of a product and also we know $\det(P^t) = \det(P)$ (see §6.4). So, combining these things we find $1 = \det(P)\det(P^t) = \det(P)^2$ and so $\det(P) = \pm 1$.]

Now

$$\det(Q) = -\det(P)$$

because we get from one to the other by swapping two rows. (See Theorem 6.4.3.) It turns out that the one with determinant $+1$ is the one that corresponds to a right handed choice of axes, and so that one is perhaps better to use.

The link with right or left handed axes (see tutorial 15, q 4) is as follows. We worked out in section 6.7 (see mention of the term 'scalar triple product') that

$$\det(P) = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$$

If we think geometrically for a while, we see that, since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are orthonormal, $\mathbf{v}\times\mathbf{w}$ is a vector in either the same or the opposite direction to $\mathbf{u}$. (Recall $\mathbf{v}\times\mathbf{w}$ is perpendicular to the plane of $\mathbf{v}$ and $\mathbf{w}$, but so also is $\mathbf{u}$.) In fact as the length of $\mathbf{v}\times\mathbf{w}$ is

$$\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin(\pi/2) = 1 = \|\mathbf{u}\|$$

we must have $\mathbf{v}\times\mathbf{w} = \pm\mathbf{u}$. For $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ to be a right handed frame, we should have $\mathbf{v}\times\mathbf{w} = \mathbf{u}$ and so

$$\det(P) = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{u}\cdot(\mathbf{u}) = \mathbf{u}\cdot\mathbf{u} = 1.$$

On the other hand if $\mathbf{v} \times \mathbf{w} = -\mathbf{u}$, then we can reverse the order to get $\mathbf{w} \times \mathbf{v} = \mathbf{u}$ and so $\det(Q) = 1$ in this case.

8.4.10 Example. Find the matrix for the rotation about the axis $\mathbf{u} = (1/\sqrt{3})(\mathbf{i} + \mathbf{j} + \mathbf{k})$ by an angle $\theta = \pi/4$, the rotation to be in the direction so that a right-handed screw placed along the axis $\mathbf{u}$ will travel in the direction of the vector $\mathbf{u}$ (rather than the opposite direction).
Solution: The idea is to come up with a right-handed frame $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, to take

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

and then (using (8.4.6)) compute

$$R = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/4) & -\sin(\pi/4) \\ 0 & \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} P$$

First we should use Gram-Schmidt to find an orthonormal frame including an axis along the direction $\mathbf{u}$. We can use $\mathbf{r} = \mathbf{i}$ and $\mathbf{s} = \mathbf{j}$. Now the steps of Gram-Schmidt.

Step 1: $\mathbf{u}$ is a unit vector already and so this step is not needed.

Just to check

$$\|\mathbf{u}\| = \frac{1}{\sqrt{3}}\|\mathbf{i} + \mathbf{j} + \mathbf{k}\|\frac{1}{\sqrt{3}}\sqrt{1^2 + 1^2 + 1^2} = 1$$

Step 2: Calculate

$$\begin{aligned}
\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u} &= \mathbf{i} - \left(\frac{1}{\sqrt{3}}\right)\mathbf{u} \\
&= \mathbf{i} - \left(\frac{1}{3}\right)(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\
&= \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \\
&= \frac{1}{3}(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \\
\|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}\| &= \frac{1}{3}\|2\mathbf{i} - \mathbf{j} - \mathbf{k}\| \\
&= \frac{1}{3}\sqrt{2^2 + (-1)^2 + (-1)^2} \\
&= \frac{\sqrt{6}}{3} \\
\mathbf{v} &= \frac{\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}}{\|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}\|} \\
&= \frac{1}{\sqrt{6}}(2\mathbf{i} - \mathbf{j} - \mathbf{k})
\end{aligned}$$

Step 3: Next compute

$$\begin{aligned}
\mathbf{s}-(\mathbf{s}\cdot\mathbf{u})\mathbf{u}-(\mathbf{s}\cdot\mathbf{v})\mathbf{v} &= \mathbf{j}-\left(\frac{1}{\sqrt{3}}\right)\mathbf{u}-\left(\frac{1}{\sqrt{6}}\right)\mathbf{v}\\
&= \mathbf{j}-\left(\frac{1}{3}\right)(\mathbf{i}+\mathbf{j}+\mathbf{k})-\left(-\frac{1}{6}\right)(2\mathbf{i}-\mathbf{j}-\mathbf{k})\\
&= \mathbf{j}-\left(\frac{1}{3}\right)(\mathbf{i}+\mathbf{j}+\mathbf{k})+\left(\frac{1}{6}\right)(2\mathbf{i}-\mathbf{j}-\mathbf{k})\\
&= \left(\frac{2}{6}-\frac{1}{3}\right)\mathbf{i}+\left(1-\frac{1}{3}-\frac{1}{6}\right)\mathbf{j}-\left(\frac{1}{3}+\frac{1}{6}\right)\mathbf{k}\\
&= \frac{1}{2}\mathbf{j}-\frac{1}{2}\mathbf{k}\\
&= \frac{1}{2}(\mathbf{j}-\mathbf{k})
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{s}-(\mathbf{s}\cdot\mathbf{u})\mathbf{u}-(\mathbf{s}\cdot\mathbf{v})\mathbf{v}\| &= \frac{1}{2}\|\mathbf{j}-\mathbf{k}\|\\
&= \frac{\sqrt{2}}{2}\\
\mathbf{w} &= \frac{\mathbf{s}-(\mathbf{s}\cdot\mathbf{u})\mathbf{u}-(\mathbf{s}\cdot\mathbf{v})\mathbf{v}}{\|\mathbf{s}-(\mathbf{s}\cdot\mathbf{u})\mathbf{u}-(\mathbf{s}\cdot\mathbf{v})\mathbf{v}\|}\\
&= \frac{1}{\sqrt{2}}((\mathbf{j}-\mathbf{k})
\end{aligned}$$

Let us check now if we have a right-handed frame from $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. The matrix P is

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and we can calculate its determinant. Using the rules for determinants it will make our lives easier if we factor out $1/\sqrt{3}$ from the first row, $2/\sqrt{6}$ from the second and $1/\sqrt{2}$ from the third.

$$\begin{aligned}
\det(P) &= \left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{2}}\right)\det\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\\ 0 & 1 & -1\end{bmatrix}\\
&= \frac{1}{\sqrt{36}}\det\begin{bmatrix}1 & 1 & 1\\ 0 & -3 & -3\\ 0 & 1 & -1\end{bmatrix} = \left(\frac{1}{6}\right)(-3)\det\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 1 & -1\end{bmatrix}\\
&= -\frac{1}{2}\det\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & -2\end{bmatrix} = -\frac{1}{2}(-2) = 1
\end{aligned}$$

Here is the same calculation done in Mathematica.

Gram Schmidt

```
u = {1, 1, 1} / Sqrt[3]
```

$$\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$$

```
r = {1, 0, 0}
```

{1, 0, 0}

```
s = {0, 1, 0}
```

{0, 1, 0}

```
v = r - (r.u) u
```

$$\left\{\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right\}$$

```
v = v / Norm[v]
```

$$\left\{\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right\}$$

```
w = s - (s.u) u - (s.v) v
```

$$\left\{0, \frac{1}{2}, -\frac{1}{2}\right\}$$

```
w = w / Norm[w]
```

$$\left\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

Change of basis matrix to orthonormal basis u, v, w

```
P = {u, v, w}
```

$$\left\{\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, \left\{\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right\}, \left\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$$

```
MatrixForm[P]
```

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
Det[P]
```

1

(Thus u, v, w is a right-handed frame.)

Now to calculate R

$$\begin{aligned}
R &= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/4) & -\sin(\pi/4) \\ 0 & \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} P \\
&= P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= P^t \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} - \frac{1}{2} & -\frac{1}{2\sqrt{3}} + \frac{1}{2} \\ \frac{2}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} + \frac{1}{2} & \frac{1}{2\sqrt{3}} - \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} - \frac{1}{2} & -\frac{1}{2\sqrt{3}} + \frac{1}{2} \\ \frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} + \frac{1}{2} & \frac{1}{2\sqrt{3}} - \frac{1}{2} \end{bmatrix}
\end{aligned}$$

As this is getting long-winded, maybe we should continue the Mathematica calculation:

■ Rotation

Rotation about x−axis by $\pi/4$

```
RR = {{1, 0, 0}, {0, Cos[Pi / 4], -Sin[Pi / 4]}, {0, Sin[Pi / 4], Cos[Pi / 4]}}
```

$$\left\{\{1, 0, 0\}, \left\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}, \left\{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}\right\}$$

```
MatrixForm[RR]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Rotation abut axis u by $\pi/4$ is given by this matrix R

```
R = Transpose[P].RR.P
```

$$\left\{\left\{\frac{1}{3}+\frac{\sqrt{2}}{3}, \frac{1}{3}-\frac{1}{3\sqrt{2}}-\frac{1}{\sqrt{6}}, \frac{1}{3}-\frac{1}{3\sqrt{2}}+\frac{1}{\sqrt{6}}\right\},\right.$$
$$\left\{\frac{1}{3}+\sqrt{\frac{2}{3}}\left(\frac{1}{2}-\frac{1}{2\sqrt{3}}\right), \frac{1}{3}-\frac{\frac{1}{2}-\frac{1}{2\sqrt{3}}}{\sqrt{6}}+\frac{\frac{1}{2}+\frac{1}{2\sqrt{3}}}{\sqrt{2}}, \frac{1}{3}-\frac{\frac{1}{2}-\frac{1}{2\sqrt{3}}}{\sqrt{6}}-\frac{\frac{1}{2}+\frac{1}{2\sqrt{3}}}{\sqrt{2}}\right\},$$
$$\left.\left\{\frac{1}{3}+\sqrt{\frac{2}{3}}\left(-\frac{1}{2}-\frac{1}{2\sqrt{3}}\right), \frac{1}{3}-\frac{-\frac{1}{2}-\frac{1}{2\sqrt{3}}}{\sqrt{6}}+\frac{-\frac{1}{2}+\frac{1}{2\sqrt{3}}}{\sqrt{2}}, \frac{1}{3}-\frac{-\frac{1}{2}-\frac{1}{2\sqrt{3}}}{\sqrt{6}}-\frac{-\frac{1}{2}+\frac{1}{2\sqrt{3}}}{\sqrt{2}}\right\}\right\}$$

Tidy up this formula:

```
R = Simplify[R]
```

$$\left\{\left\{\frac{1}{3}\left(1+\sqrt{2}\right), \frac{1}{6}\left(2-\sqrt{2}-\sqrt{6}\right), \frac{1}{6}\left(2-\sqrt{2}+\sqrt{6}\right)\right\},\right.$$
$$\left\{\frac{1}{6}\left(2-\sqrt{2}+\sqrt{6}\right), \frac{1}{3}\left(1+\sqrt{2}\right), \frac{1}{6}\left(2-\sqrt{2}-\sqrt{6}\right)\right\},$$
$$\left.\left\{\frac{1}{6}\left(2-\sqrt{2}-\sqrt{6}\right), \frac{1}{6}\left(2-\sqrt{2}+\sqrt{6}\right), \frac{1}{3}\left(1+\sqrt{2}\right)\right\}\right\}$$

```
MatrixForm[R]
```

$$\begin{pmatrix} \frac{1}{3}\left(1+\sqrt{2}\right) & \frac{1}{6}\left(2-\sqrt{2}-\sqrt{6}\right) & \frac{1}{6}\left(2-\sqrt{2}+\sqrt{6}\right) \\ \frac{1}{6}\left(2-\sqrt{2}+\sqrt{6}\right) & \frac{1}{3}\left(1+\sqrt{2}\right) & \frac{1}{6}\left(2-\sqrt{2}-\sqrt{6}\right) \\ \frac{1}{6}\left(2-\sqrt{2}-\sqrt{6}\right) & \frac{1}{6}\left(2-\sqrt{2}+\sqrt{6}\right) & \frac{1}{3}\left(1+\sqrt{2}\right) \end{pmatrix}$$

So finally we get the answer

$$R = \begin{bmatrix} \frac{1}{3}(1+\sqrt{2}) & \frac{1}{6}(2-\sqrt{2}-\sqrt{6}) & \frac{1}{6}(2-\sqrt{2}+\sqrt{6}) \\ \frac{1}{6}(2-\sqrt{2}+\sqrt{6}) & \frac{1}{3}(1+\sqrt{2}) & \frac{1}{6}(2-\sqrt{2}-\sqrt{6}) \\ \frac{1}{6}(2-\sqrt{2}-\sqrt{6}) & \frac{1}{6}(2-\sqrt{2}+\sqrt{6}) & \frac{1}{3}(1+\sqrt{2}) \end{bmatrix}$$

8.4.11 Theorem. *The 3×3 rotation matrices are exactly all the 3×3 orthogonal matrices of determinant 1.*

(This statement is also true about 2×2 matrices — see Proposition 8.3.4.)

Steps required for a proof. Perhaps we don't need to prove this in every detail, but some parts are easy enough on the basis of what we know and others are illuminating for what we will do later. The main steps are

1. Rotation matrices are orthogonal.
2. Rotation matrices have determinant 1.
3. Orthogonal matrices with determinant 1 are all rotations.

We've done the first two steps in tutorials (tutorial 15, questions 1 & 2). The last step is the hardest. We won't do it but we will show in an example some of the ingredients that go into a proof. □

8.4.12 Example. Show that the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is a rotation matrix and find the axis of rotation (up to an ambiguity of $\pm$) and the angle of rotation (up to an ambiguity between θ and $2\pi - \theta$).
Solution: Assuming we know the theorem is correct, what we have to do to establish that A is a rotation matrix is to show $AA^t = I_3$ (so A is orthogonal) and $\det(A) = 1$. Neither of these is difficult to check, but then finding the axis and the angle will be the things that involve some new techniques.

To check the two things we notice

$$AA^t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and (computing by cofactor expansion along the first row)

$$\det(A) = 0 - 1 \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 = -1(-1) = 1$$

(In fact A is an example of a permutation matrix — a single nonzero entry in each row and each column, and the nonzero entries are all $= 1$. We mentioned these permutation matrices briefly before in §6.3. In fact all of them are orthogonal matrices, but some have determinant -1.)

Now, if we want to find out which rotation has matrix A (or if we were going to try and prove the theorem) we would have to come up with a way to find out the axis for the rotation. Recall that the axis for a rotation matrix R is in a direction $\mathbf{u}$ which has $R\mathbf{u} = \mathbf{u}$. That means that the axis itself does not move. So if there is an axis for the rotation A, then we must have $A\mathbf{u} = \mathbf{u}$. We can rewrite this as

$$\begin{aligned} A\mathbf{u} &= I_3\mathbf{u} \\ (A - I_3)\mathbf{u} &= \mathbf{0} \\ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \mathbf{u} &= \mathbf{0} \\ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{u} &= \mathbf{0} \end{aligned}$$

We need a solution $\mathbf{u}$ to this which is a unit vector, or if we just find any nonzero solution $\mathbf{u}$ we can take $(1/\|\mathbf{u}\|)\mathbf{u}$ to get a unit vector. We know how to solve equations like this. What we have is a matrix form of an equation and we can solve it by row-reducing the augmented matrix

$$\begin{bmatrix} -1 & 1 & 0 & :0 \\ 0 & -1 & 1 & :0 \\ 1 & 0 & -1 & :0 \end{bmatrix}$$

First Gauss-Jordan step: multiply row 1 by -1 (to get 1 in top left).

$$\begin{bmatrix} 1 & -1 & 0 & :0 \\ 0 & -1 & 1 & :0 \\ 1 & 0 & -1 & :0 \end{bmatrix}$$

Next Gauss-Jordan step: subtract first row from third (to get new third row and zeros below leading 1 in top left).

$$\begin{bmatrix} 1 & -1 & 0 & :0 \\ 0 & -1 & 1 & :0 \\ 0 & 1 & -1 & :0 \end{bmatrix}$$

Next Gauss-Jordan step: multiply row 2 by -1 (to get leading 1 in row 2)

$$\begin{bmatrix} 1 & -1 & 0 & :0 \\ 0 & 1 & -1 & :0 \\ 0 & 1 & -1 & :0 \end{bmatrix}$$

Next Gauss-Jordan step: subtract second row from third (to get new third row and zeros below leading 1 in row 2).

$$\begin{bmatrix} 1 & -1 & 0 & :0 \\ 0 & 1 & -1 & :0 \\ 0 & 0 & 0 & :0 \end{bmatrix}$$

Next (final) Gauss-Jordan step: add second row to first (to get zeros above the last leading 1, the leading 1 in row 2).

$$\begin{bmatrix} 1 & 0 & -1 & :0 \\ 0 & 1 & -1 & :0 \\ 0 & 0 & 0 & :0 \end{bmatrix}$$

Rewrite as equations:

$$\begin{cases} u_1 & & - & u_3 & = & 0 \\ & u_2 & - & u_3 & = & 0 \end{cases}$$

So we get

$$\begin{cases} u_1 = u_3 \\ u_2 = u_3 \\ \quad\;\; u_3 \quad \text{free} \end{cases}$$

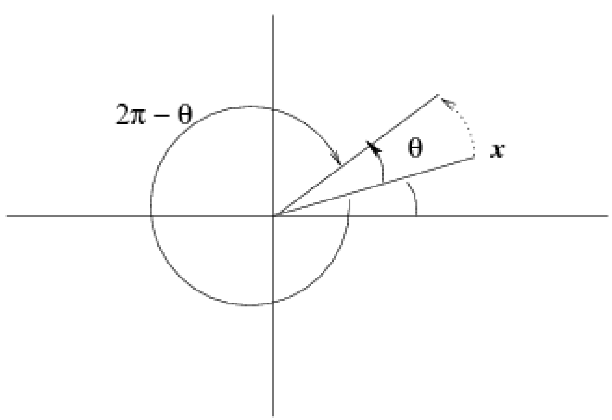
By taking a nonzero value for u_3 we get a nonzero solution. For example we could take $u_3 = 1$ which gives $u_1 = u_2 = u_3 = 1$ and the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ parallel to the axis of rotation. If we want a unit vector we should divide this by its length and get

$$\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

Is there any other possible answer for $\mathbf{u}$? Well, yes there is. If we take any positive value for u_3 we will end up with the same unit vector $\mathbf{u}$ but if we take a negative value for u_3 we end up with the unit vector in the opposite direction. So we are left with

$$\mathbf{u} = \pm\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

Which should it be? The answer is that both are possible because a rotation by an angle θ becomes an rotation by $2\pi - \theta$ when viewed upside down.



The idea of the picture is to convince you that a rotation of θ radians anticlockwise looked at from above has the same effect as a rotation by $2\pi - \theta$ radians clockwise, and the clockwise rotation of $2\pi - \theta$ will look like an anticlockwise rotation if you look at it from below (or from behind the page).

So both the plus and minus signs are possible in the vector $\mathbf{u}$ but the choice will affect the correct angle θ for the rotation.

There is a shortcut to working out the angle θ based on the trace of a matrix. We discussed the trace in §5.17 (it is the sum of the diagonal entries of a square matrix) and most of its properties are pretty easy. The only curious one was that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$ even though AB and BA need not be equal at all.

Recall from (8.4.6) that a rotation has a matrix of the form

$$R = P^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} P$$

where P is orthogonal matrix. Using the remark about traces

$$\begin{aligned}
\operatorname{trace}(R) &= \operatorname{trace}\left(P^t \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} P \right) \right) \\
&= \operatorname{trace}\left(\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} P \right) P^t \right) \\
&= \operatorname{trace}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} P P^t \right) \\
&= \operatorname{trace}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} I_3 \right) \\
&= \operatorname{trace}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \right) \\
&= 1 + \cos\theta + \cos\theta = 1 + 2\cos\theta
\end{aligned}$$

Our matrix A has

$$\operatorname{trace}(A) = \operatorname{trace}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right) = 0$$

and so the angle for our rotation has to satisfy

$$1 + 2\cos\theta = 0$$

or $\cos\theta = -1/2$. This gives two possible angles

$$\theta = \frac{2\pi}{3} \text{ or } \theta = \frac{4\pi}{3} = 2\pi - \frac{2\pi}{3}.$$

As we explained before, either choice could be right depending on whether we choose one sign or the other for the vector $\mathbf{u}$.

Unfortunately, we are still left with an ambiguity

$$\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \text{ and } \theta = \frac{2\pi}{3} \text{ or } \theta = \frac{4\pi}{3}$$

We could work out the matrix for both of these rotations and one of them must turn out to be A. But that is quite a bit more work.

8.5 Abstract approach to linear transformations

We defined linear transformations using matrices. Given an $m \times n$ matrix A we consider $f: \mathbb{R}^n \to \mathbb{R}^m$ given by $f(\mathbf{x}) = A\mathbf{x}$. Recall that we found it handy for this to write $\mathbf{x} \in \mathbb{R}^n$ as an $n \times 1$ column matrix $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ (and the same for elements of $\mathbb{R}^m$).

We have looked at some examples and we considered the examples of rotation matrices (2×2 and 3×3) is some detail. An important aspect in the 3×3 case was the desirability of changing from the original $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ frame of reference to a different orthonormal frame (or basis) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

It is useful in association with this kind of idea to have a way of describing linear transformations without reference to any particular frame of reference.

8.5.1 Theorem. *Linear transformations $f: \mathbb{R}^n \to \mathbb{R}^m$ are exactly those functions (or transformations) that satisfy the two properties*

(i) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ *(for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$)*

(ii) $f(k\mathbf{x}) = kf(\mathbf{x})$ *(for $\mathbf{x} \in \mathbb{R}^n$, k a scalar)*

Proof. There are two things to prove here:

1. Every linear transformation $f(\mathbf{x}) = A\mathbf{x}$ satisfies (i) and (ii).

2. If we have a map that satisfies (i) and (ii) then there is some $m \times n$ matrix that gives rise to the map f.

Here is how we do them

1. This is quite easy. If $f(\mathbf{x}) = A\mathbf{x}$ then we can see

$$\begin{aligned} f(\mathbf{x} + \mathbf{y}) &= A(\mathbf{x} + \mathbf{y}) \\ &= A\mathbf{x} + A\mathbf{y} \\ &\quad \text{(by properties of matrix multiplication)} \\ &= f(\mathbf{x}) + f(\mathbf{y}) \end{aligned}$$

and

$$\begin{aligned} f(k\mathbf{x}) &= A(k\mathbf{x}) \\ &= kA\mathbf{x} \\ &\quad \text{(by properties of matrix multiplication again)} \\ &= kf(\mathbf{x}) \end{aligned}$$

2. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ satisfies (i) and (ii). Then we need to find a matrix A and show it works out.

   To explain how to do this we need to introduce vectors (or points) in $\mathbb{R}^n$ which are the obvious extension of the standard basis vectors $\mathbf{i}, \mathbf{j} \in \mathbb{R}^2$ and $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{R}^3$. To avoid running out of letters we use the same letter $\mathbf{e}$ and a subscript.

   We let

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

   where they are all $n \times 1$ matrices (or n-tuples) and the j^{th} one $\mathbf{e}_j$ has 1 in the j^{th} position, zeros elsewhere.

   The idea of these (called the 'standard basis vectors' in $\mathbb{R}^n$) is that we can write every $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ as a combination

$$\begin{aligned} \mathbf{x} &= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n \end{aligned}$$

   To find the columns of the matrix A we use

$$f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots f(\mathbf{e}_n)$$

   That is we write

$$f(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, f(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, f(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

and take

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We claim now that this choice of A works out. That is we claim that it must be true that $f(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$.

To verify this claim we first do it when $\mathbf{x}$ is any of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. That is quite easy because of the way matrix multiplication works

$$\begin{aligned} A\mathbf{e}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \\ &= \text{the first column of } A \\ &= f(\mathbf{e}_1) \end{aligned}$$

Similarly

$$A\mathbf{e}_2 = \text{ second column of } A = f(\mathbf{e}_2)$$

and we get $f(\mathbf{e}_j) = A\mathbf{e}_j$ for $j = 1, 2, \ldots, n$ this way.

Finally, for any arbitrary $\mathbf{x} \in \mathbb{R}^n$ we can write

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

and use (i) and (ii) to get

$$\begin{aligned} f(\mathbf{x}) &= f(x_1\mathbf{e}_1) + f(x_2\mathbf{e}_2) + \cdots + f(x_n\mathbf{e}_n) \\ &= x_1 f(\mathbf{e}_1) + x_2 f(\mathbf{e}_2) + \cdots + x_n f(\mathbf{e}_n) \\ &= x_1 A\mathbf{e}_1 + x_2 A\mathbf{e}_2 + \cdots + x_n A\mathbf{e}_n \\ &= A(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\ &= A\mathbf{x} \end{aligned}$$

□

8.5.2 Remark. The idea then of (i) and (ii) in the Theorem above is to give an abstract interpretation of what we mean by the word “linear” in “linear transformation”.

One consequence (i) and (ii) is that we always have $f(\mathbf{0}) = \mathbf{0}$ for a linear transformation f. That is really obvious from the matrix approach — if $f(\mathbf{x}) = A\mathbf{x}$ always, then $f(\mathbf{0}) = \mathbf{0}$.

We can also get it directly from (i) like this. From (i) we have

$$\begin{aligned} f(\mathbf{0}) &= f(\mathbf{0}+\mathbf{0}) \\ &= f(\mathbf{0}) + f(\mathbf{0}) \end{aligned}$$

Then add $(-1)f(\mathbf{0})$ to both sides of this equation to get

$$\mathbf{0} = f(\mathbf{0})$$

8.6 Rigid motions

We have looked above at rotations (fixing the origin) in $\mathbb{R}^2$ and $\mathbb{R}^3$ and we've seen that they are implemented by multiplication by orthogonal matrices of determinant equal to 1.

It is possible to show (we will not do it) that the linear transformations $f \colon \mathbb{R}^n \to \mathbb{R}^n$ that arise from orthogonal matrices are exactly those that have the distance preserving property

$$\mathrm{distance}(f(\mathbf{x}), f(\mathbf{y})) = \mathrm{distance}(\mathbf{x}, \mathbf{y})$$

or

$$\|f(\mathbf{x}) - f(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$$

(for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$).

There is a point of view (which we will not pursue) that the rotations are the result of 'rigid motions' of the plane $\mathbb{R}^2$ or space $\mathbb{R}^3$. The idea is that a rigid motion is a continuous change starting from the identity (which leaves every point $\mathbf{x}$ unchanged) to end up at some final position, but we have to do it in such a way that at every intermediate time distances are preserved. Points of the plane or of space should be moved along gradually so that distances are maintained. We insist also that the origin never moves. Then it is the case that we must end up with some rotation applied to all the points.

It is not so hard to see that rotations are in this category, as we can start with a rotation by angle zero (which moves nothing) and gradually change the angle until we end up at the angle we want to reach.

Reflections of the plane cannot be arrived at by a continuous change from the identity (while also insisting that distances are always preserved). You could flip the plane gradually around the axis to end up with a rotation of π (or 180 degrees) but that involves moving the plane out of itself, and so is not considered a rigid motion of the plane. It would be a rigid motion in space.

We will not try to justify any of these statements.

8.7 Orthogonal diagonalisation

We move on now to consider linear transformations $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ that are given by diagonal matrices in some choice of frame or orthonormal basis. (We could do this in $\mathbb{R}^2$ and it would be slightly easier, or we could do it in $\mathbb{R}^n$ for any n but that seems too abstract.)

If we start with a 3×3 diagonal matrix

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

we get a linear transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbf{x}) = A\mathbf{x}$ that has

$$f(\mathbf{i}) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 \mathbf{i}$$

and we can see similarly that

$$f(\mathbf{j}) = \lambda_2 \mathbf{j} \text{ and } f(\mathbf{k}) = \lambda_3 \mathbf{k}.$$

We want to look at linear transformations f (or 3×3 matrices A) that behave like this, not on the standard basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ but instead on some other orthonormal basis $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

8.7.1 Proposition. *Suppose we have a 3×3 matrix A, and a corresponding linear transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the rule $f(\mathbf{x}) = A\mathbf{x}$.*

Then there is an orthonormal basis $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars $\lambda_1, \lambda_2, \lambda_3$ so that

$$f(\mathbf{u}) = \lambda_1 \mathbf{u}, f(\mathbf{v}) = \lambda_2 \mathbf{v} \textit{ and } f(\mathbf{w}) = \lambda_3 \mathbf{w}$$

if only if there is an orthogonal matrix P so that

$$A = P^t \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} P$$

Idea of the proof. We'll prove some of this, though none of it is really hard.

Suppose we have the orthonormal basis $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and the scalars $\lambda_1, \lambda_2, \lambda_3$. Where do we find P?

Well it is the same change of basis matrix we used before in connection with rotations.

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

(where the rows come from $\mathbf{u}, \mathbf{v}, \mathbf{w}$). Recall that

$$P\mathbf{u} = \mathbf{i}, P\mathbf{v} = \mathbf{j}, P\mathbf{w} = \mathbf{k},$$

and

$$P^t\mathbf{i} = \mathbf{u}, P^t\mathbf{j} = \mathbf{v}, P^t\mathbf{k} = \mathbf{w}.$$

Look at the matrix

$$B = P^t \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} P$$

(which we hope to show coincides with A).

If we calculate $B\mathbf{u}$ we get

$$\begin{aligned} B\mathbf{u} &= P^t \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} P\mathbf{u} \\ &= P^t \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \mathbf{i} \\ &= P^t(\lambda_1 \mathbf{i}) \\ &= \lambda_1 P^t \mathbf{i} \\ &= \lambda_1 \mathbf{u} \\ &= A\mathbf{u} \end{aligned}$$

Similarly

$$B\mathbf{v} = A\mathbf{v} \text{ and } B\mathbf{v} = A\mathbf{v}$$

Since every $\mathbf{x} \in \mathbb{R}^3$ can be written as a combination

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u})\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\mathbf{v} + (\mathbf{x} \cdot \mathbf{w})\mathbf{w}$$

of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ we can show now quite easily that $B\mathbf{x} = A\mathbf{x}$ always.

If we use this for $\mathbf{x} = \mathbf{i}$ we find

$$\text{first column of } B = \text{first column of } A$$

and we can show the other columns must coincide by taking $\mathbf{x} = \mathbf{j}$ and $\mathbf{x} = \mathbf{k}$. So $B = A$. □

8.7.2 Observation. If

$$A = P^t \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} P$$

then $A^t = A$. (We call a matrix with this property of being equal to its own transpose a *symmetric matrix*.)

Proof. Recall that the transpose of a product is the product of the transposes taken in the reverse order. So if A is as above then

$$\begin{aligned} A^t &= P^t \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}^t (P^t)^t \\ &= P^t \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} P \\ &= A \end{aligned} \qquad \square$$

8.7.3 Theorem. *The $n \times n$ symmetric matrices A are exactly those that can be written*

$$A = P^t DP$$

for an $n \times n$ orthogonal matrix P and an $n \times n$ diagonal matrix D.

8.7.4 Remark. The proof of this theorem is going to be beyond us. One bit is very easy and we've already given it in the 'Observation'. That is we showed (at least in the 3×3 case) that if $A = P^t DP$, then $A^t = A$.

The other part is the hard part and what we'll do is explain some of the ideas that you need to know if you want to use the result. That is, we'll explain how to find P and D starting with a symmetric A. We won't give any explanation of why it is always possible (in theory) to find P and D. We'll stick mainly to the 3×3 case, and the explanation here also skips over a complication that can arise (when there are only 2 different eigenvalues, rather than the usual 3).

To some extent Proposition 8.7.1 already has the bones of what we need, but it helps to have some more terminology to explain what to do.

8.7.5 Definition. A vector $\mathbf{v} \in \mathbb{R}^n$ is called an *eigenvector* for a square $n \times n$ matrix A if

- $\mathbf{v} \neq \mathbf{0}$, and
- $A\mathbf{v}$ is a multiple of $\mathbf{v}$

So there is a scalar λ so that

$$A\mathbf{v} = \lambda\mathbf{v}$$

The number λ is called the *eigenvalue* (some people may use the name 'characteristic value') for the eigenvector $\mathbf{v}$ of A.

8.7.6 Theorem. *The eigenvalues of an $n \times n$ matrix A are exactly the solutions of the* characteristic equation *for A, which is the equation*

$$\det(A - \lambda I_n) = 0$$

Proof. If λ is an eigenvalue for A, that means there is an eigenvector $\mathbf{v}$ for the eigenvalue λ. So, $A\mathbf{v} = \lambda\mathbf{v}$, and then we can say that

$$A\mathbf{v} = \lambda I_n \mathbf{v},$$

and that can be rearranged to say

$$(A - \lambda I_n)\mathbf{v} = \mathbf{0}$$

Since $\mathbf{v}$ is an eigenvector, then $\mathbf{v}$ is not the obvious solution $\mathbf{v} = \mathbf{0}$ of that equation. If we apply Theorem 5.13.2 (b) (not to the matrix A, but to the matrix $A - \lambda I_n$) this is a way of recognising that $A - \lambda I_n$ is not invertible. (If $A - \lambda I_n$ had an inverse matrix, then the equation $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$ would have only the zero solution.)

But another way to say that $A - \lambda I_n$ is not invertible is to say

$$\det(A - \lambda I_n) = 0$$

(see Theorem 6.4.2).

This shows that if λ is an eigenvalue, then λ solves the characteristic equation.

To go the other way, that is to show that if λ solves the characteristic equation then λ must be an eigenvalue, is not a very different line of argument. So we'll skip that, as we've said enough to show the connection between the concept of an eigenvalue and the solutions of the characteristic equation. □

8.7.7 Proposition (A slightly incomplete statement)**.** *If A is a symmetric matrix then we can write $A = P^t DP$ (with P orthogonal and D diagonal) if we take*

- *D to be the diagonal matrix with the eigenvalues of A along the diagonal*
- *P to be a matrix where the rows are orthonormal eigenvectors of A for the eigenvalues (in the same order as we take the eigenvalues along the diagonal of D).*

We will not prove this. The idea is to take unit vector eigenvectors. So we find the egenvalues from the characteristic equation, and then we get eigenvectors for each eigenvalue. Next we divide the eigenvectors by their length to make them unit vectors as well as eigenvectors.

The eigenevectors are usually automatically perpendicular to one another and so orthonormal. The more complicated case is where there is more than one eigenvector for the same eigenvalue. To explain that a bit more, any nonzero multiple of an eigenvector is always another eiegenvector for the same eigenvalue. (If $A\mathbf{v} = \lambda\mathbf{v}$, then $A(2\mathbf{v}) = \lambda(2\mathbf{v})$. So $2\mathbf{v}$ is again an eigenvector for the eigenvalue λ, and we can change the factor 2 to any nonzero factor.) By 'more than one eigenvector' I mean one where there are eiegenevectors that are not just multiples of each other, and still belong to the same eigenvalue λ.

Those cases are a bit more tricky to work out. But maybe we will manage not to pick any of those more complicated examples!

8.7.8 Example. For

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

find an orthogonal matrix P and a digaonal maytrix D so that $A = P^t DP$.
Solution: We want the eigenvalues and so we should work out the characteristic equation

$$\det(A - \lambda I_2) = 0$$

We have

$$A - \lambda I_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

So we have $\det(A - \lambda I_2) = (1-\lambda)(4-\lambda) - 4 = (\lambda - 1)(\lambda - 4) - 4 = \lambda^2 - 5\lambda + 4 - 4$ and the characteristic equation works out as

$$\lambda^2 - 5\lambda = 0$$

This is a quadratic equation (as it will be for every 2×2 matrix) and so we can solve it. We can do it in this case by factoring the equation as

$$\lambda(\lambda - 5) = 0$$

So the two solutions (the two eigenvalues) are $\lambda = 0$ and $\lambda = 5$.

> *Aside:* Be careful not to divide out λ without taking account of the fact that $\lambda = 0$ is a solution.

This is only a 2×2 example, rather than 3×3 as we used for illustration before. So what we have now is the two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 5$. We should take our diagonal matrix D to be

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

but to find P we need eigenvectors.

For $\lambda = \lambda_1 = 0$ (the first eigenvalue) we want a nonzero vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ so that $A\mathbf{v} = \lambda_1 \mathbf{v}$. In this case that is $A\mathbf{v} = 0\mathbf{v}$, so $A\mathbf{v} = \mathbf{0}$. We can find that by row-reducing the augmented matrix

$$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$$

The first step is to subtract 2 times row 1 from row 2, to get the new row 2:

$$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This is already completely row reduced (reduced row echelon form) and it means that we really only have one equation

$$v_1 + 2v_2 = 0$$

(as the second row gives $0 = 0$ — so tells us nothing). We can say that $v_1 = -2v_2$ and v_2 is a free variable. If we take $v_2 = 1$ then we get $v_1 = -2$ and the nonzero solution

$$\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

What would have happened if we took a different value for the free variable v_2? If we took $v_2 = 20$ we would just get 20 times the above vector. So just a multiple of the same eigenvector, not really different.

What we do want to do is to normalise that eigenvector to get one of length 1. In vector notation we have

$$\mathbf{v} = (-2)\mathbf{i} + \mathbf{j}$$

and we want to take

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{-2\mathbf{i} + \mathbf{j}}{\sqrt{(-2)^2 + 1^2}} = \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j}) = -\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

as our normalised eigenvector.

Now, for the other eigenvalue $\lambda = \lambda_2 = 5$ we want a nonzero vecor $\mathbf{v}$ (not the same $\mathbf{v}$ as we have a moment ago) so that $A\mathbf{v} = 5\mathbf{v}$. We write that as $A\mathbf{v} = 5I_2\mathbf{v}$ or $(A - 5I_2)\mathbf{v} = \mathbf{0}$. We can see that

$$A - 5I_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

and this time we want to row reduce

$$\begin{bmatrix} -4 & 2 & :0 \\ 2 & -1 & :0 \end{bmatrix}$$

We should divide row 1 by -4 (to get 1 in the top left corner) and we have

$$\begin{bmatrix} 1 & -1/2 & :0 \\ 2 & -1 & :0 \end{bmatrix}$$

and then replace row 2 by OldRow 2 $-2 \times$ OldRow1 to get

$$\begin{bmatrix} 1 & -1/2 & :0 \\ 0 & 0 & :0 \end{bmatrix}$$

Again we have just one equation $v_1 - (1/2)v_2 = 0$. Again we should take v_2 free. This time $v_1 = (1/2)v_2$ and if we pick $v_2 = 1$ we get the eigenvector

$$\frac{1}{2}\mathbf{i} + \mathbf{j}$$

(Actually it might save us some bother to take $v_2 = 2$ instead.) We should normalise this to

$$\begin{aligned}
\frac{(1/2)\mathbf{i}+\mathbf{j}}{\|(1/2)\mathbf{i}+\mathbf{j}\|} &= \frac{(1/2)\mathbf{i}+\mathbf{j}}{\sqrt{1/4+1}} \\
&= \frac{(1/2)\mathbf{i}+\mathbf{j}}{\sqrt{5/4}} \\
&= \frac{(1/2)\mathbf{i}+\mathbf{j}}{\sqrt{5}/2} \\
&= \frac{2}{\sqrt{5}}((1/2)\mathbf{i}+\mathbf{j}) \\
&= \frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{j}
\end{aligned}$$

We now have the two eigenvectors to make into the rows of P. The result is

$$P = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Notice that the rows of P are perpendicular to one another. As we have normalised the rows to be unit vectors, we have that P is an orthogonal matrix. We do then get $A = P^t DP$ or

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^t \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

If you are doubtful that the theory really holds out (or just want to check that we made no slips in the calculations) we can verify

$$\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{5} \\ 0 & 2\sqrt{5} \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

The theory says that this should always work for a symmetric A. One thing that could possibly go wrong for the 2×2 case is that the characteristic equation (the quadratic equation to solve for the eigenvalues) might have complex roots. Well, that never will happen if A is symmetric.

Let's try a 3×3 example to see how it goes. You can probably see that the calculations are fairly long even for the 2×2 case, though we could shorten what is written above by leaving out some of the chatty bits.

8.7.9 Example. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Solution: We need to work out the characteristic equation $\det(A - \lambda I_3) = 0$ and find its solutions.

First

$$A - \lambda I_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{bmatrix}$$

We can expand the determinant along the first row

$$\begin{aligned}
\det(A - \lambda I_3) &= (2-\lambda)\det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} - 1\det\begin{bmatrix} 1 & 1 \\ 1 & 3-\lambda \end{bmatrix} + 1\det\begin{bmatrix} 1 & 2-\lambda \\ 1 & 1 \end{bmatrix} \\
&= (2-\lambda)((2-\lambda)(3-\lambda)-1) - (3-\lambda-1) \\
&\quad +1-(2-\lambda) \\
&= -(\lambda-2)((\lambda-2)(\lambda-3)-1) - (2-\lambda) - 1 + \lambda \\
&= -(\lambda-2)(\lambda^2 - 5\lambda + 6 - 1) - 3 + 2\lambda \\
&= -(\lambda-2)(\lambda^2 - 5\lambda + 5) - 3 + 2\lambda \\
&= -(\lambda^3 - 5\lambda^2 + 5\lambda - 2\lambda^2 + 10\lambda - 10) - 3 + 2\lambda \\
&= -(\lambda^3 - 7\lambda^2 + 15\lambda - 10) - 3 + 2\lambda \\
&= -(\lambda^3 - 7\lambda^2 + 13\lambda - 7)
\end{aligned}$$

Now Theorem 8.7.3 says that there have to be 3 real solutions to this (3 eigenvalues for the symmetric matrix A). However, it does not help us immediately with finding the solutions. In fact every cubic equation has at least one real root, but the formula for the roots of a cubic is too messy to be useful.

In practice the best hope for finding the roots is to use the **remainder theorem**. That says that if $\lambda = \lambda_1$ is a solution of the (polynomial) equation

$$\lambda^3 - 7\lambda^2 + 13\lambda - 7 = 0$$

then $\lambda - \lambda_1$ must divide $\lambda^3 - 7\lambda^2 + 13\lambda - 7$. The only nice way to find a solution is to hope we can spot one, and the only way to do that is to try the divisors of 7 (the constant term). Those divisors are 1, -1, 7 and -7.

For 1 we get

$$(\lambda^3 - 7\lambda^2 + 13\lambda - 7)\mid_{\lambda=1} = 1 - 7 + 13 - 7 = 0$$

and so the remainder theorem says $\lambda - 1$ divides $\lambda^3 - 7\lambda^2 + 13\lambda - 7$. We can divide it in using long division of polynomials

$$\begin{array}{r|rrrrrrr}
 & \lambda^2 & - & 6\lambda & + & 7 & & \\ \hline
\lambda - 1 & \lambda^3 & - & 7\lambda^2 & + & 13\lambda & - & 7 \\
 & \lambda^3 & - & \lambda^2 & & & & \\ \hline
 & & & -6\lambda^2 & + & 13\lambda & & \\
 & & & -6\lambda^2 & + & 6\lambda & & \\ \hline
 & & & & & 7\lambda & - & 7 \\
 & & & & & 7\lambda & - & 7 \\ \hline
 & & & & & & & 0
\end{array}$$

So we have

$$\lambda^3 - 7\lambda^2 + 13\lambda - 7 = (\lambda - 1)(\lambda^2 - 6\lambda + 7)$$

and the roots of $\lambda^3 - 7\lambda^2 + 13\lambda - 7 = 0$ (our characteristic equation) are $\lambda = 1$ and the roots of the quadratic $\lambda^2 - 6\lambda + 7$. Those roots of the quadratic are

$$\lambda = \frac{6 \pm \sqrt{6^2 - 4(7)}}{2} = \frac{6 \pm \sqrt{8}}{2} = 3 \pm \sqrt{\frac{8}{4}} = 3 \pm \sqrt{2}$$

So we now have all the eigenvalues

$$\lambda = 1, \quad \lambda = 3 + \sqrt{2} \text{ and } \lambda = 3 - \sqrt{2}$$

as required.

So we are finished what was asked.

Notice that to find P so that $A = P^t DP$ we would need to find unit eigenvectors for each of these three eigenvalues — so it would take quite a while. With the help of Mathematica, I can tell you the answers. For $\lambda = 1$, the eigenvector is

$$-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j},$$

for $\lambda = 3 + \sqrt{2}$ it is

$$\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$$

and for $\lambda = 3 - \sqrt{2}$ it is

$$-\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$$

So if

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 + \sqrt{2} & 0 \\ 0 & 0 & 3 - \sqrt{2} \end{bmatrix}$$

then $A = P^t DP$.

8.7.10 Remark. Theorem 8.7.3 is quite useful because it is so simple to apply. Symmetric matrices are easy to recognise. The fact that they can be expressed as diagonal matrices in a new orthonormal basis is useful.

In the case $n = 3$, this threorem is often called the 'Principal Axis Theorem' because of an interpretation it has in mechanics. In mechanics there is a symmetric matrix called the 'interia matrix' associated with a solid object and it has to do with rotating the object around axes through the centre of mass. The axes where there will be no wobble or vibration are the axes in the direction of the eigenvectors. The 'Principal Axis Theorem' says there are always 3 such axes (though there could be more if the object is very symmetric).

The next topic is to do something similar for matrices that are not symmetric. In this case things become more complicated because there is no longer a 'Principal Axis Theorem'.

8.8 Diagonalisable matrices

8.8.1 Definition. An $n \times n$ matrix A is called *diagonalisable* if there is an invertible matrix S and diagonal matrix D (also $n \times n$ matrices) so that

$$A = SDS^{-1}$$

The matrix A is called *orthogonally diagonalisable* if we can take S to be an orthogonal matrix (so that S corresponds to what we had as P^t before).

8.8.2 Remarks. (i) We've seen that the orthogonaly diagonalisable matrices are exactly the symmetric matrices A (those with $A = A^t$). That is what Theorem 8.7.3 tells us.

(ii) In the case where A is symmetric we take $S = P^t$ where the rows of P are normalised eigenvectors of A. That means the columns of S are an orthonormal basis made up of eigenvectors.

(iii) In general if $A = SDS^{-1}$, the columns of S must be eigenvectors for A.

If, say,

$$S = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

we don't necessarily have $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ perpendicular to each other (and there is no longer any great advantage in them being normalised to have length 1).

We could explain what is going on with $A = SDS^{-1}$ in terms of changing axes from the usual ones to axes which are parallel to the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ but this brings in ideas that are rather harder to follow. The new axes are no longer perpendcular axes and all we can say is that they are in directions that use up all 3 dimensions — so no one is in the plane of the other 2.

(iv) We can go about trying to write any square matrix as $A = SDS^{-1}$ in much the same way as we did before for the case of symmetric matrices.

Step 1 is to find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I_n) = 0$. Things can go wrong at step 1. Even for 2×2 matrices, the quadratic equation we get could have only 1 root (both roots the same) or the roots could be complex numbers.

Complex numbers should not be such a problem, and things certainly work more often if we deal with comples matrices — matrices where the entries are allowed to be complex numbers as well as real numbers. However, that would be another level of complication on top of what we have done and I don't want to tackle that. It is actually not that much more difficult, but still it seems better not to go into it for this course.

If step 1 works out *and* it happes that we get n different real eigenavlues, then we can succeed in finding D and S so that $A = SDS^{-1}$. Take D to be the digaional matrix with the eigenvalues along the diagonal and S to have the eigenvectors as its columns. (You need to take the eigenvectors in the same order at the eigenvalues.)

(v) One case where the startegy works out quite nicely is for triagnular matrices (upper or lower triangular) with all the diagonal entries different.

Take for example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

The characteristic equation $\det(A - \lambda I_3) = 0$ is very easy to solve in this case because

$$A - \lambda I_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

and so

$$\det(A - \lambda I_3) = (1-\lambda)(4-\lambda)(6-\lambda)$$

Thus the eigenvalues of the (triangular) matrix A are

$$\lambda_1 = 1, \quad \lambda_2 = 4 \text{ and } \lambda_3 = 6$$

There was nothing special about the numbers 1, 2 and 6. For triangular matrices the eigenvalues are always going to work out to be the entries along the diagonal.

To find S we need eigenvectors. So, for $\lambda = 1$ we need to row reduce $[A - \lambda I_3 : \mathbf{0}] = [A - I_3 : \mathbf{0}]$, which is

$$\begin{bmatrix} 0 & 2 & 3 & :0 \\ 0 & 3 & 5 & :0 \\ 0 & 0 & 5 & :0 \end{bmatrix}$$

Divide first row by 2:

$$\begin{bmatrix} 0 & 1 & 3/2 & :0 \\ 0 & 3 & 5 & :0 \\ 0 & 0 & 5 & :0 \end{bmatrix}$$

Row 2 $-3\times$ Row 1:

$$\begin{bmatrix} 0 & 1 & 3/2 & :0 \\ 0 & 0 & 1/2 & :0 \\ 0 & 0 & 5 & :0 \end{bmatrix}$$

Row 2 times 2:

$$\begin{bmatrix} 0 & 1 & 3/2 & :0 \\ 0 & 0 & 1 & :0 \\ 0 & 0 & 5 & :0 \end{bmatrix}$$

Row 3 $-5\times$ Row 2:

$$\left[\begin{array}{cccc} 0 & 1 & 3/2 & :0 \\ 0 & 0 & 1 & :0 \\ 0 & 0 & 0 & :0 \end{array}\right]$$

In equations we have

$$\left\{\begin{array}{rcrcl} v_2 & + & \frac{3}{2}v_3 & = & 0 \\ & & v_3 & = & 0 \end{array}\right.$$

and that means $v_3 = 0$ and $v_2 = 0$ but v_1 free. So $(v_1, v_2, v_3) = (1, 0, 0)$ makes up the first column of S.

We won't go through the similar calculations in detail but the results should be $(2, 3, 0)$ for $\lambda = 4$ and $(16, 25, 10)$ for $\lambda = 6$. (Any multiples of these vectors would work just as well.) So if

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}$$

(columns of S are the eigenvectors) we have $A = SDS^{-1}$.

(vi) There are very simple matrices that are not diagonalisable (and it does not always help to allow complex numbers). One such example is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We can work out its eigenvalues by looking at the characteristic equation, but acording to the logic of the example we have just done we know how it will turn ouit. The eigenvalues are the dioagonal entries (since A is upper triangular) and so we just get $\lambda - 0$. If you like you can say that 0 is an eigenvalue twice.

So if we could write $A = SDS^{-1}$ the diagonal matrix D has to have the eigenvalues along the diagonal. In this case that means D has to be the zero matrix. But then SDS^{-1} works outr as zero, and that is not the same as A.

So this A is **not** diagonalisable.

(vii) Now for some comments to explain some reasons that diagonalisable matrices are handy. At least it is handy if we already know S and D so that $A = SDS^{-1}$ (with D diagonal as usual).

It is very easy to calculate powers of A. Look first at A^2. We have

$$A^2 = AA = SDS^{-1}SDS^{-1} = SD(S^{-1}S)DS^{-1} = SDI_nDS^{-1} = SDDS^{-1} = SD^2S^{-1}$$

$$A^3 = A^2A = SD^2S^{-1}SDS^{-1} = SD^2(S^{-1}S)DS^{-1} = SD^2I_nDS^{-1} = SD^2DS^{-1} = SD^3S^{-1}$$

and it is not hard to see that for evert power A^n we have

$$A^n = SD^nS^{-1}$$

For small powers it is a minor convenience that powers of diagonal matrices are so easy to calculate, but for big powers this is a major saving. If (say for the 3×3 case)

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

then

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

and your calculator can find powers of numbers.

(viii) Here we discuss the **exponential of a matrix**, a square matrix.

First we recall briefly that the ordinary exponential e^x of a numerical variable x can be expressed by an infinite series

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

and sometimes it is convenient to use the notation $\exp(x)$ to mean exactly the same as e^x.

This kind of infinite sum needs to be defined using limits. So that series expression for e^x says

$$\exp(x) = e^x = \lim_{N\to\infty} 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{N!}x^N$$

A limit like this does not have to exist for every x. It is easy to see that the limit does exist for $x = 0$ and $e^0 = 1$, but for the exponential series it is known that the series converges (quite quickly in fact) for every x.

By analogy with this series for e^x, if A is an $n \times n$ matrix then we define

$$e^A = \lim_{N\to\infty} I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{N!}A^N$$

and we also write this as

$$\exp(A) = e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

(an infinite series again, but this time the terms to be added up are matrices).

It would be desirable to prove that this limit always exists, but we will not do it as it becomes quite easy only with concepts that we won't develop. What we can do is give a fairly convincing way to see that e^A makes sense if A is diagonalisable, and this also contains a good way to calculate e^A.

Suppose $A = SDS^{-1}$ is diagonalisable. We've aready seen that $A^k = SD^kS^{-1}$ for all exponents $k = 1, 2, \ldots$. So we can write

$$\begin{aligned} &I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{N!}A^N \\ &= SS^{-1} + SDS^{-1} + \frac{1}{2!}SD^2S^{-1} + \frac{1}{3!}SD^3S^{-1} + \cdots + \frac{1}{N!}SD^NS^{-1} \\ &= S(I_n + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots + \frac{1}{N!}D^N)S^{-1} \end{aligned}$$

Now

$$\lim_{N\to\infty} I_n + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots + \frac{1}{N!}D^N$$

is easy to calculate. Say we take $n = 3$ as an illustration and $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$. Then

$$\begin{aligned} &I_n + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots + \frac{1}{N!}D^N \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} + \cdots + \frac{1}{N!}\begin{bmatrix} \lambda_1^N & 0 & 0 \\ 0 & \lambda_2^N & 0 \\ 0 & 0 & \lambda_3^N \end{bmatrix} \\ &= \begin{bmatrix} 1 + \lambda_1 + \frac{1}{2!}\lambda_1^2 + \cdots + \frac{1}{N!}\lambda_1^N & 0 & 0 \\ 0 & 1 + \lambda_2 + \frac{1}{2!}\lambda_2^2 + \cdots + \frac{1}{N!}\lambda_2^N & 0 \\ 0 & 0 & 1 + \lambda_3 + \cdots + \frac{1}{N!}\lambda_3^N \end{bmatrix} \end{aligned}$$

So now we can see what the limit of this is as $N \to \infty$ and we get

$$e^D = \begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{bmatrix}$$

From this it is a short step (which we will not jutsify totally) that we can multiply the limit by S on the left and S^{-1} on the right to get

$$e^A = Se^DS^{-1} = S\begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{bmatrix}S^{-1}$$

This kind of calculation is valid for diagonalisable matrices of any size (not just 3×3 as we used for illustration).

In the next topic we will see that matrix exponentials can be used fro solving differential equations.

8.9 Linear Differential Equations

We will take a rather restricted look at differential equations, aiming for an aspect where matrices can be used effectively.

A *first order linear differential equation* with constant coefficients is an equation of the form

$$\frac{dy}{dx} - ay = b$$

where a and b are constants. The unknown in a differential equation is a function $y = y(x)$ and the reason it is called a 'differential equation' is that the equation involves not just the values of y (and possibly x) but also derivatives of the unknown function. We call this one first order because the highest derivative that occurs is the first derivative dy/dx of the unknown function. Later we will talk about second order equations which involve the second derivative d^2y/dx^2. We won't actually deal with higher order equations, but third order ones would have the third derivative in them, and so on.

Sometimes these differntial equations are called 'ordinary differential equations' (abbreviated ODE sometimes) to distinguish them from 'partial differential equations' (abbreviated PDE). Both are important for many different applications, but we will not deal with PDEs at all and what w say about ODEs is quite limited. Just to explain what a PDE is, it is an equation where the unknown is a function of more than one variable and the equation involves 'partial' derivatives of the unknown function. In case you have not heard anything about partial derivatives yet, here is a very very brief explanation.

We did discuss functions of more than one variable back at the start of this chapter, though we moved fairly quickly to vector-valued (linear) functions of a vector variable. A vector variable is the same as several scalar variables at once. As an example, we can have functions

$$y = f(x_1, x_2) = x_1^4 x_2 + 3x_1^2 x_2^2 + x_1^5 - x_2^6$$

of two variables. The partial derivative of this with respect to the x_1-variable is what you get by differentiating the right hand side with respect to x_1 while treating all the other variables as constant (in this case only the variable x_2 as there are only 2 variables). Another way to explain it is that we fix some value (constant value) for x_2 while we differentiate with respect to x_1. The notation for the partial derivative is $\frac{\partial y}{\partial x_1}$ and in the example it turns out to be

$$\frac{\partial y}{\partial x_1} = 4x_1^3 x_2 + 6x_1 x_2^2 + 5x_1^4 - 0$$

On the other hand the partial derivative with respect to x_2 is

$$\frac{\partial y}{\partial x_2} = x_1^4 + 6x_1^2 x_2 + 0 - 6x_2^5$$

This brief explanation of how to calculate them is not enough to allow you to understand the ideas around partial derivatives. But anyhow, a PDE is an equation that involves partial derivatives of an unknown function (as well as the function itself and the variables x_1, x_2 and more if there are more than 2 variables invloved).

We've said that the example equation at the start of this discussion should have constant coefficients (a restriction that is not really necessary for this case) but we also said it was called a *linear* equation. Let's explain now what we mean by the word linear and why it is used at all.

The reason
$$\frac{dy}{dx} - ay = b$$
is called linear is that the left hand side
$$\frac{dy}{dx} - ay$$
depends on the unknown function y in a linear way. What we mean my this is that if we define an operation (or transformation) on functions $y = y(x)$ by
$$Ty = \frac{dy}{dx} - ay$$
then the transformation T has the same key properties we had in Theorem 8.5.1. That is

- $T(y + z) = T(y) + T(z)$ (if $y = y(x)$ and $z = z(x)$ are functions); and
- $T(ky) = kT(y)$ (for $y = y(x)$ a function and k a constant).

Recall that we first introduced linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^m$ as arising from an $m \times n$ matrix A by
$$f(\mathbf{x}) = A\mathbf{x}.$$
Later we showed in Theorem 8.5.1 that the two 'linearity' properties were a way to distinguish linear transformations from more general functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$. What we had in Theorem 8.5.1 was a more abstract way of distnguishing linear transformations, one that is not tied to the use of the standard coordinates, or the standard basis for $\mathbb{R}^n$ and $\mathbb{R}^m$.

If you look in the book, you can see that there is a more abstact approach where one can replace $\mathbb{R}^n$ and $\mathbb{R}^m$ by 'vector spaces' where you can add 'vectors' and multiply them by scalars. In this more abstract approach, our
$$Ty = \frac{dy}{dx} - ay$$
would fit right in as just one example, and we would have a more complete explantion of why we call it 'linear'.

8.9.1 Method of integrating factors. We now explain how to find all the solutions of first order linear equations
$$\frac{dy}{dx} - ay = b$$
by a method called *integrating factors*. In fact the method can be applied even if a and b are not constants but are allowed to be functioms of x, but it is particularly easy to use for the case

where a and b are constants. For the situation where we are going to bring matrices into our considerations, the restriction to constant coefficients will be more crucial We now explain how to find all the solutions of first order linear equations

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The method is to multiply the equation by

$$e^{\int -a\, dx} = e^{-ax}.$$

When we do that we get

$$e^{-ax}\frac{dy}{dx} - ae^{-ax}y = be^{-ax}$$

and the whole point of this trick is that the left hand side is now the derivative of a product. From the product rule we have

$$\frac{d}{dx}\left(e^{-ax}y\right) = e^{-ax}\frac{dy}{dx} - ae^{-ax}y$$

and so the equation we now have can be rewritten

$$\frac{d}{dx}\left(e^{-ax}y\right) = be^{-ax}$$

So we get

$$e^{-ax}y = \int be^{-ax}\, dx = -\frac{b}{a}e^{-ax} + C$$

with C some constant. Multiply now by e^{ax} on both sides to get

$$y = -\frac{b}{a} + Ce^{ax}$$

8.9.2 Example. Find all solutions of

$$\frac{dy}{dx} - 5y = 3$$

Solution: Multiply by the integrating factor

$$e^{\int -5\, dx} = e^{-5x}$$

to get

$$\begin{aligned} e^{-5x}\frac{dy}{dx} - 5e^{-5x}y &= 3e^{-5x} \\ \frac{d}{dx}\left(e^{-5x}y\right) &= 3e^{-5x} \\ e^{-5x}y &= \int 3e^{-5x}\,dx \\ &= -\frac{3}{5}e^{-5x} + C \\ y &= -\frac{3}{5} + Ce^{5x} \end{aligned}$$

Note that the solution involves a constant C which can be anything.

8.9.3 Remark. We say that we have found the general solution y for the differential equation $\frac{dy}{dx} - 5y = 3$. In an application, where we would want to know y precisely, we need some more information to pin down y.

A fairly typical case is a case where we know one value of y in addition to the fact that y satisfies the differential equation. This kind of problem is called an 'initial value problem'. An example would be to find y given that $\frac{dy}{dx} - 5y = 3$ and $y(0) = 0$. We found above that y has to have the general form $y = -\frac{3}{5} + Ce^{5x}$ for some constant C, but then we can plug in $x = 0$ to see that

$$0 = y(0) = -\frac{3}{5} + Ce^0 = -\frac{3}{5} + C$$

and that tell us that $C = 3/5$. The solution to the initial value problem is then

$$y = -\frac{3}{5} + \frac{3}{5}e^{5x}.$$

8.9.4 Remark. A consequence of linearity of equations like

$$\frac{dy}{dx} - ay = b \tag{8.9.1}$$

is that there is a relationship between solutions of this equation and solutions of the associated 'homegeneous equation' where the right hand side is replaced by 0. That is with

$$\frac{dy}{dx} - ay = 0 \tag{8.9.2}$$

The relationship can be expressed in a few ways. One way is this. If we take one solution $y = y_0$ for (8.9.1) (the inhomogeneous equation), we know

$$\frac{dy_0}{dx} - ay_0 = b \tag{8.9.3}$$

Now subtract (8.9.3) from (8.9.1) and rearrange the result using linearity to get

$$\frac{d}{dx}(y - y_0) - a(y - y_0) = 0$$

What this says is that two solutions of the inhomogeneous equation (8.9.1) have a difference $y - y_0$ that is a solution of the associated homogeneous equation (8.9.2). Another way to say that is that if we somehow know one 'particular solution' y_0 for (8.9.1), then the general solution y for (8.9.1) is

$$y = y_0 + (\text{general solution of homogeneous equation (8.9.2)})$$

This gives strategy for solving linear equations. It is not really so useful for these first order linear equations because the method of integrating factors just works out all the solutions, but it helps a lot with second order linear problems (which we will come to soon).

The strategy is this:

- somehow find one 'particular' solution for the inhomogeneous equation (by guesswork or systematic guesswork this can often be done);
- look for the general solution of the associated homogeneous equation (with 0 on the right):

  (This can be easier to deal with than having to cope with the right hand side at the same time.)
- the general solution of the inhomegeneous equation is then

$$(\text{particular solution}) + (\text{general solution of homogeneous equation})$$

8.9.5 Second Order Linear. We now move on to second order linear differential equations. We will only deal with the case of constant coefficients (which makes things much easier) and we will also discuss only the homogeneous case. The strategy we have just outlined about 'particular' solutions + general solutions of the associated homegeneous equation is a very common way to approach these second order problems. So we will be dealing with most of the significant issues by looking at the homogeneous case.

That means we will discuss equations of the type

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where b and c are constants. We could allow for a constant coefficient in front of the second derivative term. But, we can divide acrorss by that coefficient to get an equation like the one above. (Well we can do that if we are not dividing by 0. If the coefficient of d^2y/dx^2 was 0, then we would have a first order problem, not a second order one.)

Now there is a trick to reduce to a first order problem, but at the expense of getting two equations (a system of equations).

The trick is to introduce a new name y_1 for y and a temporary

$$y_2 = \frac{dy}{dx} = \frac{dy_1}{dx}$$

Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy_2}{dx}$. We can then rewrite the second order equation as

$$\frac{dy_2}{dx} + by_2 + cy_1 = 0$$

but we need to keep in mind the connection between y_2 and y_1 also. We get a system

$$\begin{cases} \frac{dy_1}{dx} = y_2 \\ \frac{dy_2}{dx} + by_2 + cy_1 = 0 \end{cases}$$

or

$$\begin{cases} \frac{dy_1}{dx} = y_2 \\ \frac{dy_2}{dx} = -cy_1 - by_2 \end{cases}$$

Using matrices we can write this system as a single equation between two column matrices

$$\begin{bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{bmatrix} = \begin{bmatrix} y_2 \\ -cy_1 - by_2 \end{bmatrix}$$

and using matrix multiplication we can write that as

$$\begin{bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Next we treat the unknown in this equation as a vector valued function

$$\mathbf{y} = \mathbf{y}(x) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and we agree to say that differnetiating such a function means differentiating each component. That is we define

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{bmatrix}$$

Now we can write our matrix equation (which came about from the system of two equations) as a differential equation for the vector-valued unknown $\mathbf{y}(x) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. We get

$$\frac{d\mathbf{y}}{dx} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \mathbf{y}$$

If we use A to stand for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$$

then our equation is

$$\frac{d\mathbf{y}}{dx} = A\mathbf{y}$$

To make it look even more like what we had before we write this

$$\frac{d\mathbf{y}}{dx} - A\mathbf{y} = \mathbf{0} \tag{8.9.4}$$

This looks so temptingly similar to the first order linear (homogeneous) equation

$$\frac{dy}{dx} - ay = 0$$

that we might like to try and use an integrating factor method to solve the new version. When a is a scalar we multiplied by the integrating factor $e^{\int -a\,dx} = e^{-ax}$ and so maybe we should multiply across by e^{-Ax}.

Since we conveniently discussed exponentials of matrices already, we can makes sense of e^{-Ax}. The order of matrix multiplication matters a lot and we multiply $\frac{d\mathbf{y}}{dx} - A\mathbf{y} = \mathbf{0}$ by e^{-Ax} on the left. As e^{-Ax} is a 2×2 matrix and the equation is an equality of 2×1 column matrices, we have to multiply on the left. We get

$$e^{-Ax}\frac{d\mathbf{y}}{dx} - e^{-Ax}A\mathbf{y} = e^{-Ax}\mathbf{0}$$

or

$$e^{-Ax}\frac{d\mathbf{y}}{dx} - e^{-Ax}A\mathbf{y} = \mathbf{0}$$

It turns out to be the case that

$$\frac{d}{dx}e^{-Ax} = -e^{-Ax}A$$

if we interpret differentiation of a matrix-valued function of x to mean differentiation of each entry seperately.

Why is that true? Well we are not really in a position to show it in general, but it is easy enough to see that it works out when A is replaced by a diagonal matrix D. If $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ then

$$e^{-Dx} = \exp\left(\begin{bmatrix} -\lambda_1 x & 0 \\ 0 & -\lambda_2 x \end{bmatrix}\right) = \begin{bmatrix} e^{-\lambda_1 x} & 0 \\ 0 & e^{-\lambda_2 x} \end{bmatrix}$$

So

$$\begin{aligned}\frac{d}{dx}e^{-Dx} &= \begin{bmatrix}\frac{d}{dx}e^{-\lambda_1 x} & 0\\ 0 & \frac{d}{dx}e^{-\lambda_2 x}\end{bmatrix}\\ &= \begin{bmatrix}-\lambda_1 e^{-\lambda_1 x} & 0\\ 0 & -\lambda_2 e^{-\lambda_2 x}\end{bmatrix}\\ &= -\begin{bmatrix}e^{-\lambda_1 x} & 0\\ 0 & e^{-\lambda_2 x}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = -e^{-Dx}D\end{aligned}$$

It is possible to come up with a modification of this argument to show $\frac{d}{dx}e^{-Ax} = -e^{-Ax}A$ when $A = SDS^{-1}$ is diagonalisable, but actually it is true for every square matrix A.

Using $\frac{d}{dx}e^{-Ax} = -e^{-Ax}A$ we can rewrite our equation (8.9.4) as

$$e^{-Ax}\frac{d\mathbf{y}}{dx} + \frac{d}{dx}\left(e^{-Ax}\right)\mathbf{y} = \mathbf{0} \tag{8.9.5}$$

What we need now is a product rule for differentiating matrix products. We will not check it works, but it is true that if $U = U(x)$ and $V = V(x)$ are matrix functions so that the matrix product UV makes sense, then

$$\frac{d}{dx}(UV) = \left(\frac{dU}{dx}\right)V + U\frac{dV}{dx}$$

It is important to keep the order here so that U always stays to the left of V.

Using this product rule, we can rewrite (8.9.5) as

$$\frac{d}{dx}\left(e^{-Ax}\mathbf{y}\right) = \mathbf{0}$$

and it should be clear that the only vector functions that have derivative $\mathbf{0} = \begin{bmatrix}0\\0\end{bmatrix}$ are constants.

So we get

$$e^{-Ax}\mathbf{y} = \text{ constant vector } = \begin{bmatrix}c_1\\c_2\end{bmatrix}$$

To find $\mathbf{y}$, we multiply by the inverse matrix of e^{-Ax}, which turns out to be e^{Ax}. We get

$$\mathbf{y} = \begin{bmatrix}y_1\\y_2\end{bmatrix} = e^{Ax}\begin{bmatrix}c_1\\c_2\end{bmatrix} \tag{8.9.6}$$

as the solution of our system of differential equations (8.9.4).

Of course we should work this out further so it does not involve the matrix exponential. Let us assume (to make our life easier) that we are always in the case where A is diagonalisable. So we can write $A = SDS^{-1}$ with $D = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$ diagonal. Then

$$Ax = SDS^{-1}x = S(Dx)S^{-1}$$

(because x is a scalar) and $Dx = \begin{bmatrix} \lambda_1 x & 0 \\ 0 & \lambda_2 x \end{bmatrix}$ diagonal. We have then

$$e^{Ax} = Se^{Dx}S^{-1} = S \begin{bmatrix} e^{\lambda_1 x} & 0 \\ 0 & e^{\lambda_2 x} \end{bmatrix} S^{-1}$$

When we go to use this in (8.9.6) the first thing will be to multiply out

$$S^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

It will make life easier for us if we just write the result as

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = S^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

for two new constants α_1 and α_2. So now out solution (8.9.6) comes to

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{Ax} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S \begin{bmatrix} e^{\lambda_1 x} & 0 \\ 0 & e^{\lambda_2 x} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = S \begin{bmatrix} \alpha_1 e^{\lambda_1 x} & 0 \\ 0 & \alpha_2 e^{\lambda_2 x} \end{bmatrix}$$

Recall now, finally, that the columns of S are eigenvectors for A belonging to the eigenvalues λ_1 and λ_2. So S has the form

$$S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors written as columns. That means that the solution works out as

$$\mathbf{y} = \alpha_1 e^{\lambda_1 x} \mathbf{v}_1 + \alpha_2 e^{\lambda_2 x} \mathbf{v}_2$$

We now summarise what all these calculations show:

8.9.6 Theorem. *Assume that* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *is a diagonalisable matrix, with eigenvalues* λ_1 *and* λ_2 *and corresponding eigenvectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$. *Then the solutions to the system of linear differential equations*

$$\begin{cases} \dfrac{dy_1}{dx} = a_{11}y_1 + a_{12}y_2 \\ \dfrac{dy_2}{dx} = a_{21}y_1 + a_{22}y_2 \end{cases}$$

are given by

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \alpha_1 e^{\lambda_1 x} \mathbf{v}_1 + \alpha_2 e^{\lambda_2 x} \mathbf{v}_2$$

where α_1, α_2 *are arbitrary constants.*

8.9.7 Example. We did start with a second order equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

and we rewrote it as a system

$$\frac{d\mathbf{y}}{dx} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \mathbf{y}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ dy/dx_2 \end{bmatrix}$$

According to the above, we need the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$$

So we should look at the characteristic equation $\det(A - \lambda I_2) = 0$. We have

$$A - \lambda I_2 = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -c & -b - \lambda \end{bmatrix}$$

and so

$$\det(A - \lambda I_2) = -\lambda(-b - \lambda) + c = \lambda^2 + b\lambda + c$$

Notice then the close similarity between the differential equation and the characteristic equation. Replace the second derivative term by λ^2, the derivative term by λ and the y term by 1.

To be specific we take the example

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$$

so the the characteristic equations is

$$\lambda^2 - 5\lambda + 4 = 0.$$

This factors as $(\lambda - 1)(\lambda - 4) = 0$ and so the eigenvalues are $\lambda = 1$ and $\lambda = 4$.

According to the recipe we also need the eigenvectors. We might say $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ for $\lambda = 1$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ for $\lambda = 4$. Then we get

$$\begin{bmatrix} y \\ \frac{dy}{dx} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \alpha_1 e^x \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \alpha_2 e^{4x} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

We don't really need to work out the eigenvectors in this case because we can see that

$$y = \alpha_1 v_1 e^x + \alpha_2 w_1 e^{4x}$$

and we can just regard $\alpha_1 v_1$ and $\alpha_2 w_1$ as some constants. We get

$$y = C_1 e^x + C_2 e^{4x}$$

for constants C_1 and C_2, where $\lambda_1 = 1$ and $\lambda_2 = 4$ are the roots of the equation $\lambda^2 - 5\lambda + 4 = 0$ that comes from the equation.

It is worth pointing out though that we have ignored the possibility that the quadratic might have complex roots (or just one root).

8.9.8 Example. Here is a somewhat applied example.

Two competing species live on the same small island and each one affects the growth rate of the other (by competing for the same food, say). If thir populations at time t are $x_1(t)$ and $x_2(t)$, a model for their growth rates says

$$\begin{cases} x_1'(t) &= -3x_1(t) + 6x_2(t) \\ x_2'(t) &= x_1(t) - 2x_2(t) \end{cases}$$

At time $t = 0$, $x_1(0) = 500$ and $x_2(0) = 200$.

Find $x_1(t)$ and $x_2(t)$.

Solution: We can write the model in matrix form as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

We need the eigenvalues and eigenvectors for $A = \begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix}$ in order to write down the general solution of this system of first order linear differential equations (and later we need to use the information about $x_1(0)$ and $x_2(0)$ to find the constants).

We have

$$A - \lambda I_2 = \begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3-\lambda & 6 \\ 1 & -2-\lambda \end{bmatrix}$$

and so

$$\det(A - \lambda I_2) = (-3-\lambda)(-2-\lambda) - 6 = (\lambda+3)(\lambda+2) - 6 = \lambda^2 + 5\lambda + 6 - 6 = \lambda(\lambda+5)$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -5$.

To find the eigenvalue for $\lambda_1 = 0$ we should row reduce $[A : \mathbf{0}]$. The answer should be the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -5$, we have $A - \lambda_2 I_2 = A + 5I_2 = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ and we should row reduce

$$\begin{bmatrix} 2 & 6 & :0 \\ 1 & 3 & :0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & :0 \\ 1 & 3 & :0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & :0 \\ 0 & 0 & :0 \end{bmatrix}$$

So that eigenvector is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

The general solution is then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \alpha_1 e^0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2 e^{-5t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 - 3\alpha_2 e^{-5t} \\ \alpha_1 + \alpha_2 e^{-5t} \end{bmatrix}$$

If we put $t = 0$ we are supposed to get

$$\begin{bmatrix} 500 \\ 200 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2\alpha_1 - 3\alpha_2 \\ \alpha_1 + \alpha_2 \end{bmatrix}$$

We then have 2 simultaneous equations to solve for α_1 and α_2. We could set that up as a matrix to be row reduced, but anyhow the solution is $\alpha_1 = 220$, $\alpha_2 = -20$.

So the answer to the example is

$$\begin{aligned} x_1(t) &= 440 + 60e^{-5t} \\ x_2(t) &= 220 - 20e^{-5t} \end{aligned}$$

(Aside: I'm not sure how realistic the model was since the value of $x_2(t)$ becomes neagtive after some time t, and populations can't be a negative number.)

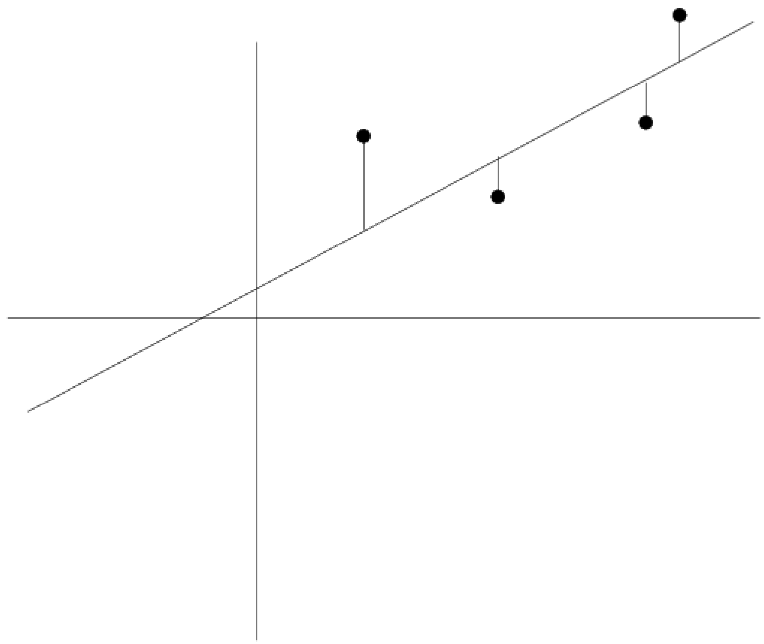
8.10 Least square fit

We now discuss a topic that is rather important for applications.

Suppose we do an experiment and find some data points that we know (from the theory of the experiment) are supposed to lie on a line. Say we found n data points (x_1, y_1), (x_2, y_2), $\ldots (x_n, y_n)$ and they are supposed to lie on a line $y = mx + c$. How do we find the right line?

If we had two points we would just find the line through the two points, but we will often have more than two data obersvations and (unless we fiddled the data) it is unlikely they will be on any one line. What then is the 'best' estimate of the line?

Well, that does depend on what you mean by 'best' but the least squares approach is often considered to be an appropriate interpretation of best. What it means is to choose the line that makes the sum of the square of the vertical distance from the data points to the line as small as possible. Here is a picture that is supposed to show the idea. The blobs represent 4 data points and the line should be so that the sum of the squares of the vertical segments joining the blobs to the line is smaller than for any other line.



This is not the only interpretation of 'best' that could be considered. It is reasonably appropriate if the data is such that the x-values $x_1, x_2, \ldots, x_n$ are correct while the experimental errors are in the measurements of the y-values $y_1, y_2, \ldots, y_n$. The idea is that the 'correct' y-values are $y_1^*, y_2^*, \ldots, y_n^*$ and that these are so that the 'corrected' data points

$$(x_1, y_1^*), (x_2, y_2^*), \ldots, (x_n, y_n^*)$$

lie on the 'correct' line. Moreover the most likely line is the one where we should make the smallest overall correction to the data — that is the one that make

$$(y_1 - y_1^*)^2 + (y_2 - y_2^*)^2 + \cdots + (y_n - y_n^*)^2$$

as small as possible.

To make the notation more convenient we will rewrite the equation of the line as $y = \beta_0 + \beta_1 x$ (so β_0 is the constant term and β_1 is the coefficient of x or the slope). If all the data points were on a line, it would be the line that solved all the equations

$$\begin{aligned} \beta_0 + \beta_1 x_1 &= y_1 \\ \beta_0 + \beta_1 x_2 &= y_2 \\ &\vdots \\ \beta_0 + \beta_1 x_n &= y_n \end{aligned}$$

In matrix form we can write this as

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

or

$$X\boldsymbol{\beta} = \mathbf{y}$$

where we use the notation

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

As we said already, once $n > 2$ we have too many equations for just 2 unknowns β_0 and β_1. There is very little chance that there is any solution for $\boldsymbol{\beta}$. The approach is to modify $\mathbf{y}$ to get $\mathbf{y}^*$ so that we do get a solution to

$$X\boldsymbol{\beta} = \mathbf{y}^*$$

and

$$\|\mathbf{y} - \mathbf{y}^*\|^2 = (y_1 - y_1^*)^2 + (y_2 - y_2^*)^2 + \cdots + (y_n - y_n^*)^2$$

is as small as possible.

We'll take $n = 3$ for the purpose of explaining how that is done. We need some understanding of where we are allowed to take $\mathbf{y}^*$ so as to make $X\boldsymbol{\beta} = \mathbf{y}^*$ solvable. We can write

$$X\boldsymbol{\beta} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta_1 \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$$

What we can say then is that as β_0 and β_1 change we always find that $X\boldsymbol{\beta}$ lies in the plane in space $\mathbb{R}^3$ through the origin and containing the two vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

So the best choice of $\mathbf{y}^*$ will be the point in that plane you get by dropping a perpendicular from $\mathbf{y}$ onto the plane. (That will give the point $\mathbf{y}^*$ on the plane closest to $\mathbf{y}$.)

We can go about calculating that point, but there is a trick to avoid that. The difference $\mathbf{y} - \mathbf{y}^*$ should be a vector perpendicular to the plane. So $\mathbf{y} - \mathbf{y}^*$ is perpendicular to both $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and to $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$. We can write that in matrix notation as

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{bmatrix} (\mathbf{y} - \mathbf{y}^*) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

and that is just $X^t(\mathbf{y} - \mathbf{y}^*) = \mathbf{0}$. So we find that

$$X^t\mathbf{y} = X^t\mathbf{y}^*$$

So instead of finding $\mathbf{y}^*$ and then solving

$$X\boldsymbol{\beta} = \mathbf{y}^*$$

we multiply this equation by X^t on the left. That gives

$$X^t X\boldsymbol{\beta} = X^t \mathbf{y}^*$$

which is the same as

$$X^t X\boldsymbol{\beta} = X^t \mathbf{y}$$

Now $X^t X$ is just a 2×2 matrix and $X^t\mathbf{y}$ is a 2×1 matrix. We are down to 2 equations in 2 unknowns β_0 and β_1. They are known as the *normal equations*.

Summary: To find the line $y = \beta_0 + \beta_1 x$ that is the best least squares fit to the data points

$$(x_1, y_1^*), (x_2, y_2^*), \ldots, (x_n, y_n^*)$$

solve the normal equations

$$X^t X\boldsymbol{\beta} = X^t \mathbf{y}$$

where

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

8.10.1 Example. Find the equation of the line that is the best least squares fit to the data points

$$(2, 1), (5, 2), (7, 3), (8, 3)$$

Solution: We take

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

and solve the normal equations $X^t X\boldsymbol{\beta} = X^t\mathbf{y}$.

We need to calculate

$$X^t X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}$$

and

$$X^t\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

And now solve

$$\begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

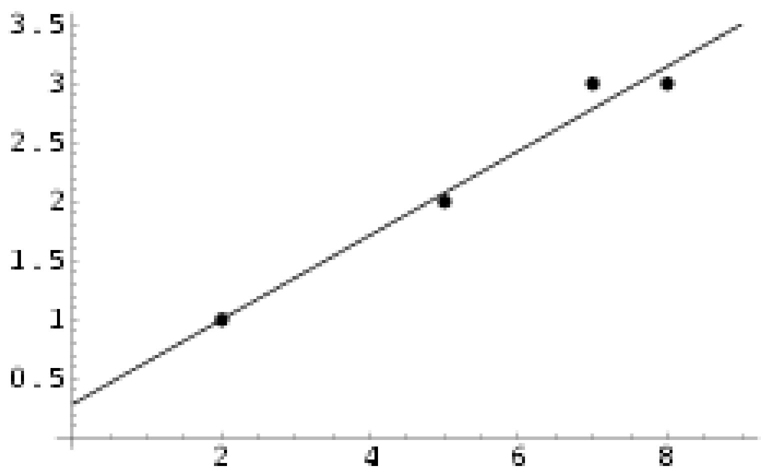
We can do that by row reducing

$$\left[\begin{array}{ccc} 4 & 22 & : 9 \\ 22 & 142 & : 57 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 11/2 & : 9/4 \\ 22 & 142 & : 57 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 11/2 & : 9/4 \\ 0 & 21 & : 15/2 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 11/2 & : 9/4 \\ 0 & 1 & : 5/14 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 0 & : 2 \\ 0 & 1 & : 5/ \end{array}\right.$$

So $\beta_0 = 2/7$ and $\beta_1 = 5/14$. The line is

$$y = \frac{2}{7} + \frac{5}{14}x$$

and here is a picture of the line with the points



8.11 Markov Matrices

One thing we could have mentioned earlier is that a matrix always has the same eigenvalues as its transpose. If A is an $n \times n$ (square) matrix we do know (from 6.4 (iv)) that $\det(A^t) = \det(A)$. It follows quite easily that A and A^t have the same chacteristic equation. The reason is that

$$A^t - \lambda I_n = A^t - \lambda I_n^t = (A - \lambda I_n)^t$$

and so $\det(A^t - \lambda I_n) = \det(A - \lambda I_n)$. So the chacteristic equation $\det(A^t - \lambda I_n) = 0$ is the same equation as $\det(A - \lambda I_n) = 0$.

One place where this little fact is useful is in studying Markov matrices. Markov matrices are square matrices which have

- all entries ≥ 0, and
- in each column, the sum of the entries in the column is 1

This is a 3×3 Markov matrix

$$A = \begin{bmatrix} 0.2 & 0 & 0.4 \\ 0.2 & 0.5 & 0.1 \\ 0.6 & 0.5 & 0.5 \end{bmatrix}$$

If we take the transpose of this Markov matrix and multiply it by the column of all 1's, we find

$$A^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0 & 0.5 & 0.1 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(because the row sums in the transpose are all equal 1).

What this shows is that if A is a Markov matrix, then the column of all 1's is an eigenvector of A^t with the eigenvalue 1. So there must be an eigenvector for the matrix A with the eigenvalue 1, that is a vector $\mathbf{v}$ with

$$A\mathbf{v} = \mathbf{v}$$

That vector $\mathbf{v}$ fixed by A is rather important in considerations of Markov matrices. There are many applications of Markov matrices (for example in finding what happens with genetic types over many generations) but we will stop our study of linear algebra in this module here, without looking into these interesting topics!

TO BE checked