Chapter 1. Linear Equations

We'll start our study of linear algebra with linear equations. Lost of parts of mathematics rose out of trying to understand the solutions of different types of equations. Linear equations are probably the simplest kind.

1.1 What is linear and not linear

Here are some examples of equations that are maybe interesting from some point of view

$$x^2 + 3x - 4 = 0 \quad (1)$$
$$x^2 + 3x + 4 = 0 \quad (2)$$
$$x^2 = 2 \quad (3)$$
$$x - \sin x = 1 \quad (4)$$

but **none of these are linear!**

They are perhaps noteworthy for different reasons. Equation (1) is easy to solve by factorisation. Equation (2) is harder — it can't be factored and if you use the quadratic formula to get the solutions, you get complex roots:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 16}}{2} = \frac{-3 \pm \sqrt{-7}}{2} = \frac{-3 \pm \sqrt{7}\sqrt{-1}}{2}.$$

So the need for complex numbers emerged from certain quadratic equations. The equation (3) is not that complicated but it was upsetting for the Greek mathematicians of a few thousand years ago to realise that there are no solutions that can be expressed as $x = \frac{p}{q}$ with p and q whole numbers. (Such numbers are called *rational numbers* and $\sqrt{2}$ is an *irrational number* because it is not such a fraction.) Perhaps it is not so obvious that $\sin x$ is not linear, but it is not.

1.2 Linear equations with a single unknown

So what *are* linear equations? Well they are very simple equations like

$$3x + 4 = 0.$$

There is not much to solving this type of equation (for the unknown x). For reasons we will see later, let's explain the simple steps involved in solving this example in a way that may seem unnecessarily long-winded. Add -4 to both sides of the equation to get

$$3x = -4.$$

Then multiply both sides of this by $\frac{1}{3}$ (or divide both sides by 3 if you prefer to put it like that) to get

$$x = -\frac{4}{3}.$$

What we need to understand about these simple steps is that they transform the problem (of solving the equation) to a problem with all the same information. If x solves $3x + 4 = 0$ then it has to solve $3x = -4$. But on the other hand, we can go back from $3x = -4$ to the original $3x + 4 = 0$ by adding $+4$ to both sides. In this way we see that any x that solves $3x = -4$ must also solve $3x + 4 = 0$. So the step of adding -4 is reversible. Similarly the step of multiplying by $\frac{1}{3}$ is reversible by multiplying by 3.

1.3 Linear equations with two unknowns

Well, that little explanation seems hardly necessary, and how could we be having a course about such a simple thing? We can start to make things a little more complicated if we introduce a linear equation in 2 unknowns, like

$$5x - 2y = 1. \tag{5}$$

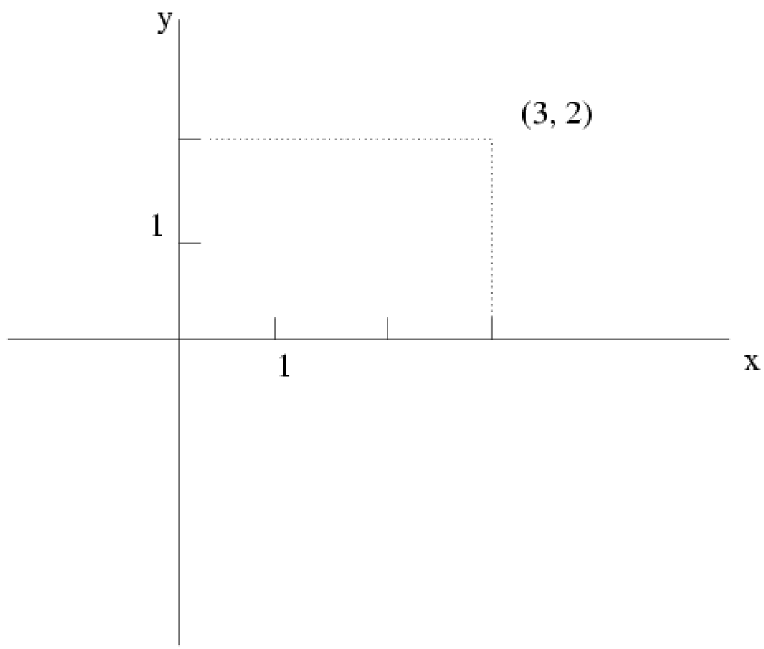
(Instead of having unknowns called x and y we could instead call have two unknowns called x_1 and x_2, and an equation like $5x_1 - 2x_2 = 1$. This would be more or less the same.)

What are the solutions to equation (5)? We can't really "solve" it because one equation is not enough to find 2 unknowns. What we can do is solve for y in terms of x.

$$\begin{aligned} 5x - 2y &= 1 \\ -2y &= -5x + 1 & \text{(add } -5x \text{ to both sides)} \\ y &= \frac{5}{2}x - \frac{1}{2} & \text{(multiply both sides by } -\tfrac{1}{2}) \end{aligned}$$

When we look at it like this we can understand things in at least 2 ways. We are free to choose any value of x as long as we take $y = \frac{5}{2}x - \frac{1}{2}$. So we have infinitely many solutions for x and y. Another way is to think graphically. $y = \frac{5}{2}x - \frac{1}{2}$ is an equation of the type $y = mx + c$, the equation of a line in the x-y plane with slope $m = \frac{5}{2}$ that crosses the y-axis at the place where $y = c = -\frac{1}{2}$.

Thus we can think of points in a plane as labeled by two coordinates (x, y) once we fix two perpendicular axes. The axes need a scale and a direction. Usually we think of the x-axis as the horizontal one, pointing to the right, and the y-axis as (vertical and) pointing upwards.



The solutions of a single linear equation in 2 unknowns can be visualised as the set of all the points on a straight line in the plane.

Going back to the mechanism of solving, we could equally solve $5x - 2y = 1$ for x in terms of y. We'll write that out because it is actually the way we will do things later. (We will solve for the variable listed first in preference to the one listed later.)

$$\begin{aligned} 5x - 2y &= 1 \\ x - \frac{2}{5}y &= \frac{1}{5} \text{ (multiply both sides by } \tfrac{1}{5}) \\ x &= \frac{1}{5} + \frac{2}{5}y \qquad \text{(add } \tfrac{2}{5}y \text{ to both sides)} \end{aligned}$$

We end up with y a *free variable* and once we choose y we get a solution as long as we take $x = \frac{1}{5} + \frac{2}{5}y$.

1.4 Systems of linear equations

If we now move to systems of equations (also known as simultaneous equations) where we want to understand the (x, y) that solve all the given equations simultaneously, we can have examples like

$$\left\{ \begin{array}{rcrcl} 5x & - & 2y & = & 1 \\ 3x & + & 4y & = & 8 \end{array} \right. \tag{6}$$

or

$$\left\{ \begin{array}{rcrcl} 5x & - & 2y & = & 1 \\ 3x & + & 4y & = & 8 \\ 26x & - & 26y & = & -17, \end{array} \right. \tag{7}$$

we can think about the problem graphically. One linear equation describes a line and so the solutions to the system (6) should be the point (or points) where the two lines meet. The solutions to (7) should be the point (or points) where the three lines meet. Here is a graph (drawn by Mathematica) of the lines (6)

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In[1]:= Plot[{ (5 / 2) x - 1 / 2,  -3 / 4 x +2}, {x, -3, 3}]
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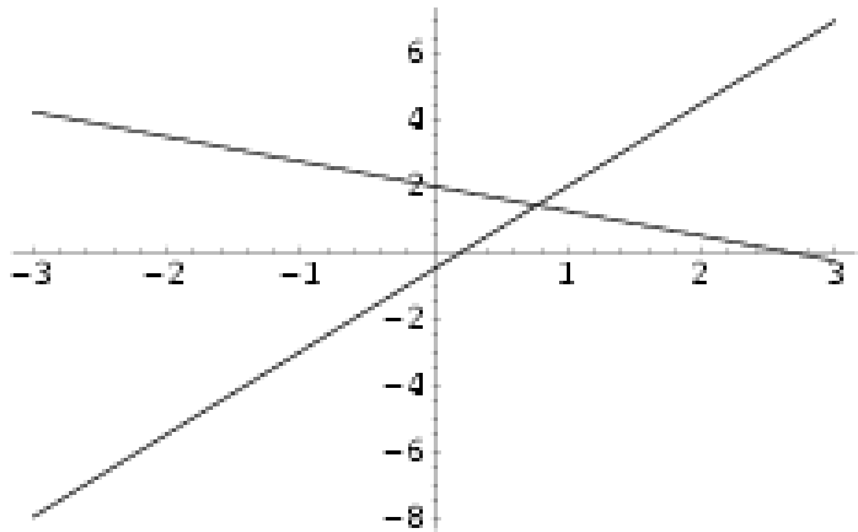


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Out[1]= -Graphics-
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and this is the picture for (7)

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In[2]:= Plot[{ (5 / 2) x - 1 / 2,  -3 / 4 x +2, y = x + 17 / 26}, {x, -3, 3}]
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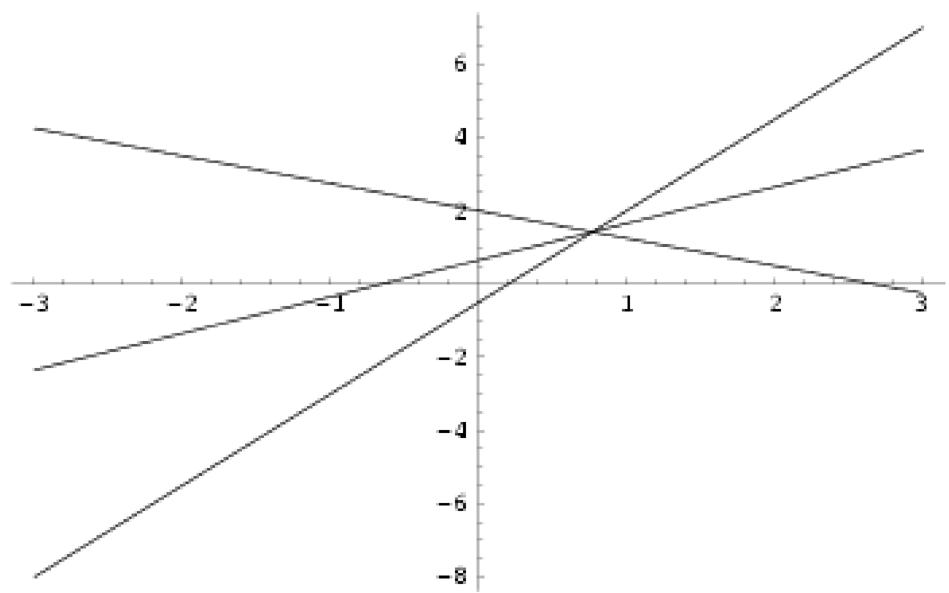


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Out[2]= -Graphics-
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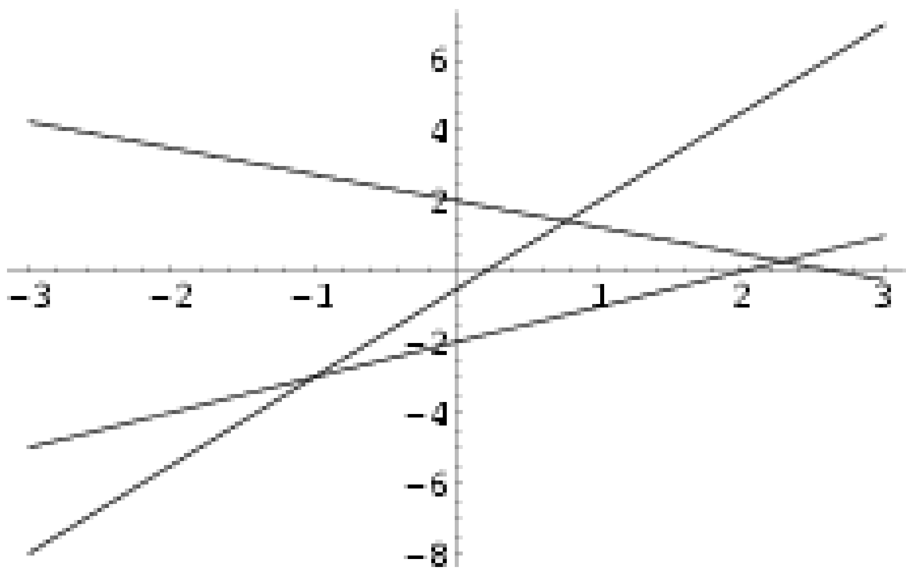
You can see that, in this particular case, the two systems (6) and (7) have the same solution. There is just one solution $x = 10/13$, $y = 37/26$, or one point $(x, y) = (10/13, 37/26)$ on all the lines in (6). Same for (7).

However, you can also start to see what can happen in general. If you take a system of two linear equations in two unknowns, you will typically see two lines that are not parallel and they will meet in one point. But there is a chance that the two lines are parallel and never meet, or a chance that somehow both equations represent the same line and so the solutions are all the points on the line.

When you take a 'typical' system of 3 equations in 2 unknowns, you should rarely expect to see the kind of picture we just had. It is rather a fluke that 3 lines will meet in a point. Typically they will look like



and will not meet in a point. You can see geometrically or graphically what is going on when you solve systems of equations in just 2 unknowns

If we move to equations in 3 unknowns

$$\left\{\begin{array}{rcrcrcr} 5x & - & 2y & + & z & = & 4 \\ 3x & + & 4y & + & 4z & = & 10 \\ 26x & - & 26y & - & z & = & -7, \end{array}\right. \tag{8}$$

it is still possible to visualise what is going on. However it require thinking in space, 3 dimensions, and we are less ready to do that. A bit later we should be able to do that and to see that the solutions of a single linear equation in 3 variables (like $5x - 2y + z = 4$) can be visualised as the points on a (flat) plane in space. The common solutions of two such equations will usually be the points on the line of intersection of the two planes (unless the planes are parallel). Typically, 3 planes will meet in a single point (but there is a chance they all go through one line).

This idea of visualisation is good for understanding what sort of things can happen, but it is not really practical for actually calculating the solutions. Worse than that, we are pretty much sunk when we get to systems of equations in 4 or more unknowns. They need to be pictured in a space with at least 4 dimensions. And that is not practical.

Here is an example of a single linear equation in 4 unknowns x_1, x_2, x_2 and x_4

$$5x_1 - 2x_2 + 6x_3 - 7x_4 = 15$$

1.5 Solving systems of equations, preliminary approach

We turn instead to a recipe for solving systems of linear equations, a step-by-step procedure that can always be used. It is a bit harder to see what the possibilities are (about what can possibly happen) and a straightforward procedure is a valuable thing to have.

We'll explain with an example:

$$\left\{ \begin{array}{rcrcrcrcr} 3x_1 & + & x_2 & - & x_3 & + & x_4 & = & 5 \\ 15x_1 & + & 5x_2 & + & x_3 & & & = & 7 \\ 9x_1 & + & 3x_2 & & & - & x_4 & = & 0 \end{array} \right.$$

Our goal is to solve the system as completely as possible and to explain what all the solutions are in a way that is as clear and simple as possible.

Step 1 Make first equation start with $1x_1$.

In our example we do this by the step

$$\text{NewEquation1} = (\text{OldEquation1}) \times \frac{1}{3}, \text{ other equations unchanged}$$

and the result of this is

$$\left\{ \begin{array}{rcrcrcrcr} x_1 & + & \frac{1}{3}x_2 & - & \frac{1}{3}x_3 & + & \frac{1}{3}x_4 & = & \frac{5}{3} \\ 15x_1 & + & 5x_2 & + & x_3 & & & = & 7 \\ 9x_1 & + & 3x_2 & & & - & x_4 & = & 0 \end{array} \right.$$

Step 2 Eliminate x_1 from all but the first equation.

We do this by subtracting appropriate multiples of the first equation from each of the others in turn. So, in this case, we do

$$\begin{array}{rcl} \text{NewEquation1} & = & \text{OldEquation1}, \\ \text{NewEquation2} & = & \text{OldEquation2} - 15(\text{OldEquation1}), \\ \text{NewEquation3} & = & \text{OldEquation3} - 9(\text{OldEquation1}). \end{array}$$

and the result of that is

$$\left\{ \begin{array}{rcrcrcrcr} x_1 & + & \frac{1}{3}x_2 & - & \frac{1}{3}x_3 & + & \frac{1}{3}x_4 & = & \frac{5}{3} \\ 0 & + & 0 & + & 6x_3 & - & 5x_4 & = & -18 \\ 0 & + & 0 & + & 5x_3 & - & 4x_4 & = & -15 \end{array} \right.$$

Step 3 Leave equation 1 as it is for now (it is now the only one with x_1 in it). Manipulate the remaining equations as in step 1, but taking account of only the remaining variables x_2, x_3 and x_4.

In our example, since we can't get $1x_2$ from 0, we must skip x_2 and concentrate on x_3 instead. Next step is

$$\text{NewEquation2} = (\text{OldEquation2}) \times \frac{1}{6}, \text{ other equations unchanged}$$

We get

$$\left\{\begin{array}{rcrcrcrcr} x_1 & + & \frac{1}{3}x_2 & - & \frac{1}{3}x_3 & + & \frac{1}{3}x_4 & = & \frac{5}{3} \\ 0 & + & 0 & + & x_3 & - & \frac{5}{6}x_4 & = & -3 \\ 0 & + & 0 & + & 5x_3 & - & 4x_4 & = & -15 \end{array}\right.$$

Next, repeating a step like step 2, to eliminate x_3 from the equations below the second, we do

$$\text{NewEquation3} = \text{OldEquation3} - 3(\text{OldEquation2})$$

to get

$$\left\{\begin{array}{rcrcrcrcr} x_1 & + & \frac{1}{3}x_2 & - & \frac{1}{3}x_3 & + & \frac{1}{3}x_4 & = & \frac{5}{3} \\ 0 & + & 0 & + & x_3 & - & \frac{5}{6}x_4 & = & -3 \\ 0 & + & 0 & + & 0 & - & \frac{3}{2}x_4 & = & -6 \end{array}\right.$$

Next we do step 1 again, ignoring both the first two equations now (and putting aside x_3).

$$\text{NewEquation3} = (\text{OldEquation3}) \times \left(-\frac{2}{3}\right)$$

and we have

$$\left\{\begin{array}{rcrcrcrcr} x_1 & + & \frac{1}{3}x_2 & - & \frac{1}{3}x_3 & + & \frac{1}{3}x_4 & = & \frac{5}{3} \\ 0 & + & 0 & + & x_3 & - & \frac{5}{6}x_4 & = & -3 \\ 0 & + & 0 & + & 0 & - & x_4 & = & 4 \end{array}\right.$$

Step 4 is called back-substitution.

The last equation tells us what x_4 has to be ($x_4 = 4$). We regard the second equation as telling us what x_3 is and the first telling us what x_1 is. Rewriting them we get

$$\left\{\begin{array}{lcl} x_1 & = & \frac{5}{3} - \frac{1}{3}x_2 + \frac{1}{3}x_3 - \frac{1}{3}x_4 \\ x_3 & = & -3 + \frac{5}{6}x_4 \\ x_4 & = & 4 \end{array}\right.$$

There is a further simplification we can make. Using the last equation in the ones above we get

$$\left\{\begin{array}{lcl} x_1 & = & \frac{5}{3} - \frac{1}{3}x_2 + \frac{1}{3}x_3 - \frac{4}{3} \\ & = & \frac{1}{3} - \frac{1}{3}x_2 + \frac{1}{3}x_3 \\ x_3 & = & -3 + \frac{10}{3} \\ & = & \frac{1}{3} \\ x_4 & = & 4 \end{array}\right.$$

Finally, now that we know x_3 we can plug that into the equation above

$$\begin{cases} x_1 &= \frac{1}{3} - \frac{1}{3}x_2 + \frac{1}{9} = \frac{4}{9} - \frac{1}{3}x_2 \\ x_3 &= \frac{1}{3} \\ x_4 &= 4 \end{cases}$$

The point of the steps we make is partly that the system of equations we have at each stage has exactly the same solutions as the system we had at the previous stage. We already tried to explain this principle with the simplest kind of example in 1.2 above.

If x_1, x_2, x_3 and x_4 solve the original system of equations, then they must solve all the equations in the next system because we get the new equations by combining the original ones. But all the steps we do are reversible by a step of a similar kind. So any x_1, x_2, x_3 and x_4 that solve the new system of equations must solve the original system.

What we have ended up with is a system telling us what x_1, x_3 and x_4 are, but nothing to restrict x_2. We can take any value for x_2 and get a solution as long as we have

$$x_1 = \frac{4}{9} - \frac{1}{3}x_2, x_3 = \frac{1}{3}, x_4 = 4.$$

We have solved for x_1, x_3 and x_4 in terms of x_2, but x_2 is then a *free variable*.

1.6 Solving systems of equations, Gaussian elimination

We will now go over the same ground again as in 1.5 but using a new and more concise notation.

Take an example again

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 5 \\ x_1 & & & + & x_3 & = & -2 \\ 2x_1 & + & x_2 & + & 3x_3 & = & 4 \end{array}$$

In the last example, we were always careful to write out the equations keeping the variables lined up (x_1 under x_1, x_2 under x_2 and so on). As a short cut, we will write only the numbers, that is the coefficients in front of the variables and the constants from the right hand side. We write them in a rectangular array like this

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 3 & 4 \end{array} \right]$$

and realise that we can get the equations back from this table because we have kept the numbers in such careful order.

A rectangular table of numbers like this is called a *matrix* in mathematical language. When talking about matrices (the plural of matrix is matrices) we refer to the *rows*, *columns* and *entries* of a matrix. We refer to the rows by number, with the top row being row number 1. So in the above matrix the first row is

$$[1 \quad 1 \quad 2 \quad 5]$$

Similarly we number the columns from the left one. So column number 3 of the above matrix is

$$\begin{matrix} 2 \\ 1 \\ 3 \end{matrix}$$

The entries are the numbers in the various positions in the matrix. So, in our matrix above the number 5 is an entry. We identify it by the row and column it is on, in this case row 1 and column 4. So the $(1,4)$ entry is the number 5.

In fact, though it is not absolutely necessary now, we often make an indication that the last column is special. We mark the place where the $=$ sign belongs with a dotted vertical line, like this:

$$\begin{bmatrix} 1 & 1 & 2 & : & 5 \\ 1 & 0 & 1 & : & -2 \\ 2 & 1 & 3 & : & 4 \end{bmatrix}$$

A matrix like this is called an *augmented matrix*.

The process of solving the equations, the one we used in 1.5 can be translated into a process where we just manipulate the rows of an augmented matrix like this. The steps that are allowed are called *elementary row operations* and these are what they are

(i) multiply all the numbers is some row by a nonzero factor (and leave every other row unchanged)

(ii) replace any chosen row by the difference between it and a multiple of some other row.

[This really needs a little more clarification. What we mean is to replace each entry in the chosen row by the same combination with the corresponding entry of the 'other' row.]

(iii) Exchange the positions of some pair of rows in the matrix.

We see that the first two types, when you think of them in terms of what is happening to the linear equations, are just the sort of steps we did with equations above. The third kind, row exchange, just corresponds to writing the same list of equations in a different order. We need this to make it easier to describe the step-by-step procedure we are going to describe now.

The procedure is called *Gaussian elimination* and we aim to describe it in a way that will always work. Moreover, we will describe it so that there is a clear recipe to be followed, one that does not involve any choices. Here it is.

Step 1 Organise top left entry of the matrix to be 1.

How:

- if the top left entry is already 1, do nothing
- if the top left entry is a nonzero number, multiply the first row across by the reciprocal of the top left entry
- if the top left entry is 0, but there is some nonzero entry in the first column, swap the position of row 1 with the first row that has a nonzero entry in the first column; then proceed as above
- if each entry in the first column is 0, ignore the first column and move to the next column; then proceed as above

Step 2 Organise that the first column has all entries 0 below the top left entry.

How: subtract appropriate multiples of the first row from each of the rows below in turn

Step 3 Ignore the first row and first column and repeat steps 1, 2 and 3 on the remainder of the matrix until there are either no rows or no columns left.

Once this is done, we can write the equations for the final matrix we got and solve them by back-substitution.

We'll carry out this process on our example, in hopes of explaining it better

$$\left[\begin{array}{ccccc} 1 & 1 & 2 & : & 5 \\ 1 & 0 & 1 & : & -2 \\ 2 & 1 & 3 & : & 4 \end{array}\right]$$

$$\left[\begin{array}{ccccc} 1 & 1 & 2 & : & 5 \\ 0 & -1 & -1 & : & -7 \\ 0 & -1 & -1 & : & -6 \end{array}\right| \begin{array}{l} \\ \text{OldRow2} - \text{OldRow1} \\ \text{OldRow3} - 2(\text{OldRow1}) \end{array}$$

That is step 1 and step 2 done (first time). Just to try and make it clear, we now close our eyes to the first row and column and work essentially with the remaining part of the matrix.

$$\left[\begin{array}{c|cccc} 1 & 1 & 2 & : & 5 \\ \hline 0 & -1 & -1 & : & -7 \\ 0 & -1 & -1 & : & -6 \end{array}\right| \begin{array}{l} \\ \text{OldRow2} - \text{OldRow1} \\ \text{OldRow2} - 2(\text{OldRow1}) \end{array}$$

Well, we don't actually discard anything. We keep the whole matrix, but we look at rows and columns after the first for our procedures. So what we do then is

$$\left[\begin{array}{ccccc} 1 & 1 & 2 & : & 5 \\ 0 & 1 & 1 & : & 7 \\ 0 & -1 & -1 & : & -6 \end{array}\right| \begin{array}{l} \\ \text{OldRow2} \times (-1) \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|l} 1 & 1 & 2 & : & 5 & \\ 0 & 1 & 1 & : & 7 & \\ 0 & 0 & 0 & : & 1 & \text{OldRow3} + \text{OldRow2}) \end{array}\right.$$

That is steps 1 and 2 completed again (now with reference to the smaller matrix).

We should now cover over row 2 and column 2 (as well as row 1 and column 1) and focus our process on what remains. All that remains in $0 \quad 1$. We kip the column of zeros and the next thing is 1. So we are done. That is we have finished the Gaussian elimination procedure.

What we do now is write out the equations that correspond to this. We get

$$\left\{\begin{array}{ccccccc} x_1 & + & x_2 & + & 2x_3 & = & 5 \\ & & x_2 & + & x_3 & = & 7 \\ & & & & 0 & = & 1 \end{array}\right.$$

As the last equation in this system is never true (no matter what values we give the unknowns) we can see that there are no solutions to the system. They are called an *inconsistent system of linear equations*.

So that is the answer. No solutions for this example.

1.7 Another example

Let's try an example that does have solutions.

$$\left\{\begin{array}{ccccccc} & & 2x_2 & + & 3x_3 & = & 4 \\ x_1 & - & x_2 & + & x_3 & = & 5 \\ x_1 & + & x_2 & + & 4x_3 & = & 9 \end{array}\right.$$

Write it as an augmented matrix.

$$\left[\begin{array}{ccccc} 0 & 2 & 3 & : & 4 \\ 1 & -1 & 1 & : & 5 \\ 1 & 1 & 4 & : & 9 \end{array}\right]$$

$$\left[\begin{array}{ccccc|l} 1 & -1 & 1 & : & 5 & \text{OldRow2} \\ 0 & 2 & 3 & : & 4 & \text{OldRow1} \\ 1 & 1 & 4 & : & 9 & \end{array}\right.$$

$$\left[\begin{array}{ccccc|l} 1 & -1 & 1 & : & 5 & \\ 0 & 2 & 3 & : & 4 & \\ 0 & 2 & 3 & : & 4 & \text{OldRow3} - \text{OldRow1} \end{array}\right.$$

$$\left[\begin{array}{ccccc|l} 1 & -1 & 1 & : & 5 & \\ 0 & 1 & \frac{3}{2} & : & 2 & \text{OldRow2} \times \left(\frac{1}{2}\right) \\ 0 & 2 & 3 & : & 4 & \end{array}\right.$$

$$\left[\begin{array}{ccccc|l} 1 & -1 & 1 & : & 5 & \\ 0 & 1 & \frac{3}{2} & : & 2 & \\ 0 & 0 & 0 & : & 0 & \text{OldRow3} - 2 \times \text{OldRow2} \end{array}\right.$$

Writing out the equations for this, we get

$$\left\{\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 5 \\ & & x_2 & + & \frac{3}{2}x_3 & = & 2 \\ & & & & 0 & = & 0 \end{array}\right.$$

or

$$\left\{\begin{array}{rcl} x_1 & = & 5 + x_2 - x_3 \\ x_2 & = & 2 - \frac{3}{2}x_3 \end{array}\right.$$

Using back-substitution, we put the value for x_2 from the second equation into the first

$$\left\{\begin{array}{rcl} x_1 & = & 5 + \left(2 - \frac{3}{2}x_3\right) - x_3 \\ & = & 7 - \frac{5}{2}x_3 \\ x_2 & = & 2 - \frac{3}{2}x_3 \end{array}\right.$$

Thus we end up with this system of equations, which must have all the same solutions as the original system. We are left with a free variable x_3, which can have any value as long as we take $x_1 = 7 - \frac{5}{2}x_3$ and $x_2 = 2 - \frac{3}{2}x_3$. So

$$\left\{\begin{array}{rcl} x_1 & = & 7 - \frac{5}{2}x_3 \\ x_2 & = & 2 - \frac{3}{2}x_3 \\ & & x_3 \text{ free} \end{array}\right.$$

describes all the solutions.

1.8 Row echelon form

The term *row echelon form* refers to the kind of matrix we must end up with after successfully completing the Gauss-Jordan method. Here are the features of such a matrix

- The first nonzero entry of each row is equal to 1 (unless the row consists entirely of zeroes). We refer to these as *leading ones*.
- The leading one of any row after the first must be at least one column to the right of the leading ones of any rows above.
- Any rows of all zeroes are at the end.

It follows that, for a matrix in row echelon form, there are zeroes directly below each lading one.

(The origin of the word *echelon* has to do with the swept back wing form made by the leading ones. Wikipedia says “Echelon formation, a formation in aerial combat, tank warfare, naval warfare and medieval warfare; also used to describe a migratory bird formation” about this. The Merriam-Webster Online Dictionary gives the first definition as: ‘an arrangement of a body of troops with its units each somewhat to the left or right of the one in the rear like a series of steps’ and goes on to mention formations of geese. Perhaps geese fly in a V formation and the echelon is one wing of the V.)

1.9 Gauss-Jordan elimination

There is an alternative to using back-substitution. It is almost equivalent to doing the back substitution steps on the matrix (before converting to equations — when we do this we will then have no more work to do on the equations as they will be completely solved). The method (or recipe) for this is called Gauss-Jordan elimination. Here is the method

Steps 1–3 Follow the steps for Gaussian elimination (resulting in a matrix that has row-echelon form).

Step 4 Starting from the bottom, arrange that there are zeros above each of the leading ones.

How:

- Starting with the last nonzero row, subtract appropriate multiples of that row from all the rows above so as to ensure that there are all zeros above the leading 1 of the last row. (There must already be zeros below it.)
- Leave that row aside and repeat the previous point on the leading one of the row above, until all the leading ones have zeroes above.

Following completion of these steps, the matrix should be in what is called *reduced row-echelon form*. That means that the matrix should be in row-echelon form **and also** the column containing each leading 1 must have zeroes in every other entry.

Let us return to the example in 1.7 and finish it by the method of Gauss-Jordan. We had the row-echelon form

$$\left[\begin{array}{ccccc} 1 & -1 & 1 & : & 5 \\ 0 & 1 & \frac{3}{2} & : & 2 \\ 0 & 0 & 0 & : & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc} 1 & 0 & \frac{7}{2} & : & 7 \\ 0 & 1 & \frac{3}{2} & : & 2 \\ 0 & 0 & 0 & : & 0 \end{array}\right] \begin{array}{l} \text{OldRow1} + \text{OldRow2} \\ \\ \\ \end{array}$$

and we already have the matrix in reduced row-echelon form.

When we write out the equations corresponding to this matrix, we get

$$\left\{\begin{array}{ccccccc} x_1 & & & + & \frac{7}{2}x_3 & = & 7 \\ & & x_2 & + & \frac{3}{2}x_3 & = & 2 \\ & & & & 0 & = & 0 \end{array}\right.$$

and the idea is that each equation tells us what the variable is that corresponds to the leading 1. Writing everything but that variable on the right we get

$$\left\{\begin{array}{ccc} x_1 & = & 7 - \frac{5}{2}x_3 \\ x_2 & = & 2 - \frac{3}{2}x_3 \end{array}\right.$$

and no equation to tell us x_3. So x_3 is a free variable.