Mathematics 121 2004–05
Exercises 4
[Due Wednesday January 26th, 2005.]

1. Recall that we define right-hand limits $\lim_{x\to a^+} f(x)$ in case the domain of f includes an open interval (a,b) with $b > a$ and in this situation the limit is ℓ if and only if

   For each $\varepsilon > 0$ there exists $\delta > 0$ so that
   $$0 < x - a < \delta \Rightarrow |f(x) - \ell| < \varepsilon$$

   Prove that the following sequence criterion is equivalent to $\lim_{x\to a^+} f(x) = \ell$:

   For each sequence $(x_n)_{n=1}^\infty$ in the domain of f with $x_n > a$ for all n and $\lim_{n\to\infty} x_n = a$, it is true that
   $$\lim_{n\to\infty} f(x_n) = \ell.$$

2. Recall: The left-hand limit $\lim_{x\to a^-} f(x) = \ell$ means the domain of f includes an open interval (c,a) with $c < a$ and

   For each $\varepsilon > 0$ there exists $\delta > 0$ so that
   $$0 < a - x < \delta \Rightarrow |f(x) - \ell| < \varepsilon$$

   Assuming that $f\colon S \to \mathbb{R}$ is a function defined on $S \subset \mathbb{R}$ where S contains an open interval to the left of a, prove that the following sequence criterion is equivalent to $\lim_{x\to a^+} f(x) = \ell$:

   For each sequence $(x_n)_{n=1}^\infty$ in the domain of f with $x_n < a$ for all n and $\lim_{n\to\infty} x_n = a$, it is true that
   $$\lim_{n\to\infty} f(x_n) = \ell.$$

3. Assuming that $f\colon S \to \mathbb{R}$ is a function defined on $S \subset \mathbb{R}$ where S contains a punctured open interval about $a \in \mathbb{R}$ show that
   $$\lim_{x\to a} f(x) = \ell \iff \lim_{x\to a^+} f(x) = \ell \text{ and } \lim_{x\to a^-} f(x) = \ell$$

   [Hint: The $\Leftarrow$ direction is probably easiest using ε-δ.]

4. Assuming that $f\colon S \to \mathbb{R}$ is a function defined on $S \subset \mathbb{R}$ where S contains a punctured open interval about $a \in \mathbb{R}$ show that $\lim_{x\to a} f(x)$ exists if and only if both one-sided limits ($\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$) exist and
   $$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$$

   [Hint: Use the previous problem.]

5. If $S \subset \mathbb{R}$ is a compact set and $f\colon S \to \mathbb{R}$ is continuous, show that the range $f(S) = \{f(x) : x \in S\}$ is also a compact set.

   (In summary 'continuous images of compact sets are compact.')

6. If $S \subset \mathbb{R}$ is compact show that S is bounded.

7. If $S \subset \mathbb{R}$ is compact and non-empty, show that $\text{lub}(S) \in S$.