Mathematics 121 2004–05
Exercises 3
[Due Wednesday December 8th, 2004.]

1. Show, by directly using the ε-N definition of limits of sequences, that the following statements are correct:

   (a) $\displaystyle\lim_{n\to\infty} 1 + \frac{(-1)^n}{n+1} = 1$

   (b) $\displaystyle\lim_{n\to\infty} \frac{n^3 - 5n^2 + 2n + 7}{4n^3 + n - 1} = \frac{1}{4}$

2. Use the theorem on limits of sums, products and quotients to give another proof of the statement in part (b) of question 1.

3. Prove that if $(x_n)_{n=1}^{\infty}$ is a bounded sequence and $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty} x_n y_n = 0$.

4. Give an example of two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ where $\lim_{n\to\infty}(x_n + y_n)$ exists but

$$\lim_{n\to\infty}(x_n + y_n) \neq \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

5. If a sequence $(x_n)_{n=1}^{\infty}$ has integer terms ($x_n \in \mathbb{Z}$ for all n) and $\lim_{n\to\infty} x_n = \ell \in \mathbb{R}$, what can you say about ℓ and x_n for n large?

6. If $f\colon \mathbb{R} \to \mathbb{R}$ is bounded (meaning that the range $f(\mathbb{R}) = \{f(x) : x \in \mathbb{R}\}$ is bounded) show that

$$\lim_{x\to 0} x f\left(\frac{1}{x}\right) = 0.$$

7. Find an exmple of a function $f\colon \mathbb{R} \to \mathbb{R}$ where

$$\lim_{x\to 0} x f\left(\frac{1}{x}\right)$$

   is not 0.