Mathematics 121 2004–05
Exercises 2
[Due Friday November 26th, 2004.]

1. Prove the following assertions based on the axioms (P1) - (P12) for $\mathbb{R}$ and the consequences of them we have checked earlier:

   (a) $|x \cdot y| = |x| \cdot |y|$ (for $x, y \in \mathbb{R}$)

   (b) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$ (for $x \in \mathbb{R}$, $x \neq 0$)

2. Use the triangle inequality to show that if $a, b, c \in \mathbb{R}$ then the following are true:

   (a) $|a - b| \geq |a| - |b|$

   (b) $|a - b| \geq ||a| - |b||$

   (c) $|a + b + c| \leq |a| + |b| + |c|$

3. Find the least upper bound and the greatest lower bound of each of the following sets (or show that they do not exist):

   (a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

   (b) $\left\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\right\}$

   (c) $\left\{x \in \mathbb{R} : x = 0 \text{ or } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}\right\}$

   (d) $\left\{x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2}\right\}$

   Also, in each case, decide if there is a least and/or a greatest element in the set.

4. Show that if $x \in \mathbb{R}$ and $x \geq 0$, then there is $n \in \mathbb{N}$ with $n - 1 \leq x < n$. [Hint: If this is false, a proof by induction that $n \leq x$ for all $n \in \mathbb{N}$ would work.]

5. Show that if $x \in \mathbb{R}$, then there is $n \in \mathbb{Z}$ with $n \leq x < n + 1$. [Hint: Try $x \geq 0$ first.]

6. Show that every $x \in \mathbb{R}$ with $0 \leq x < 1$ has a decimal expansion $x = 0.d_1d_2d_3\ldots = \lim_{n\to\infty} \sum_{j=1}^{n} d_j 10^{-j}$ where each $d_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. [Hint: Choose d_1 so that $d_1 \leq 10x < d_1 + 1$.]