Mathematics 121 2004–05
Exam
[Tuesday, December 14, 9.30 — 11.00]

Attempt 3 questions.

1. **(a)** State the 4 axioms satisfied by the operation $+$ (addition) on the real numbers $\mathbb{R}$.

   Solution: Reproduce notes (P1) - (P4) (Chapter 1)

   (b) Show that each real number x has a unique additive inverse.

   Solution: This is Lemma 1.4 in the notes.

   (c) Show (using only the axioms for $+$) that the following cancellation law holds for additive equations: $a, b, c \in \mathbb{R}, a + c = b + c \Rightarrow a = b$.

   Solution: Adding the additive inverse $-c$ of c to both sides we get from $a+c = b+c$ that

   $$(a + c) + (-c) = (b + c) + (-c)$$

   and rearranging both sides with the associative law for addition this gives

   $$a + (c + (-c)) = b + (c + (-c))$$

   The definition of an additive inverse implies that $c + (-c) = 0$ and so we have

   $$a + 0 = b + 0$$

   from which we get

   $$a = b$$

   by the defining property of 0.

2. **(a)** Define the term *upper bound* for a subset of $\mathbb{R}$ and the term *least upper bound*. Also state the least upper bound principle.

   Solution: See Definition 1.16 and (P13) in the notes

   (b) Prove that the natural numbers are not bounded above in $\mathbb{R}$.

   Solution: See proof of Proposition 1.18(i) in the notes.

   (c) Give an example of a non-empty set of rational numbers which is bounded above but has no least upper bound in $\mathbb{Q}$. Justify your example.

   Solution: One example is $S = \{x \in \mathbb{Q} : 0 < x \text{ and } x^2 < 2\}$.

   Notice that $S \subset \mathbb{Q}$ as required ("set of rational numbers") and $S \neq \emptyset$ because $1 \in S$. S is bounded above by 2. To see this suppose on the contrary $x \in S$ and $x > 2$. Then $x^2 > 2x > 2^2 = 4 > 2$, contradicting $x^2 < 2$. We conclude that $x \leq 2$ for each $x \in S$, that is that 2 is an upper bound for S.

   Note that $S \subset \mathbb{Q} \subset \mathbb{R}$ and the least upper bound principle implies that there is a least upper bound $u \in \mathbb{R}$ for S. Our claim is that $u \notin \mathbb{Q}$.

If $u \in \mathbb{Q}$ then we know $u^2 \neq 2$ as $\sqrt{2} \notin \mathbb{Q}$, and thus there are two other possibilities (i) $u^2 < 2$ and (ii) $u^2 > 2$. We now show that neither (i) nor (ii) are the case.

If $u^2 < 2$ (and $u \in \mathbb{Q}$), we claim that there is $n \in \mathbb{N}$ with $u + \frac{1}{n} \in S$ (and this would contradict u being an upper bound). Now

$$\left(u + \frac{1}{n}\right)^2 = u^2 + 2\frac{u}{n} + \frac{1}{n^2} \leq u^2 + 2\frac{u}{n} + \frac{1}{n} = u^2 + \frac{2u+1}{n}.$$

It follows that $\left(u + \frac{1}{n}\right)^2 < 2$ if $u^2 + \frac{2u+1}{n} < 2$ or, equivalently if $\frac{2u+1}{n} < 2 - u^2$. Since $u \geq 1 \in S$, we have $u > 0$ and so $2u + 1 > 0$. Thus $\frac{2u+1}{n} < 2 - u^2$ is equivalent to $\frac{1}{n} < \frac{2-u^2}{2u+1}$. We can find an $n \in \mathbb{N}$ with this property by the proposition above. For such $n \in \mathbb{N}$ we then have $\left(u + \frac{1}{n}\right)^2 < 2$ and $u + \frac{1}{n} > u > 0$. Hence $u + \frac{1}{n} \in S$ contradicting u an upper bound for S. So (i) is eliminated.

If $u^2 > 2$ we claim that there is $n \in \mathbb{N}$ so that $u - \frac{1}{n}$ is an upper bound for S (smaller than the least upper bound S and so a contradiction). We choose $n \in \mathbb{N}$ so that $\left(u - \frac{1}{n}\right)^2 > 2$. This we can do because

$$\left(u - \frac{1}{n}\right)^2 = u^2 - 2\frac{u}{n} + \frac{1}{n^2} > u^2 - 2\frac{u}{n}$$

and this will be > 2 we ensure

$$u^2 - 2\frac{u}{n} > 2$$

or, equivalently

$$u^2 - 2 > 2\frac{u}{n}.$$

We can arrange this by choosing $n \in \mathbb{N}$ (via the proposition above) so that or $\frac{1}{n} < \frac{u^2-2}{2u}$. This is possible because $u > 0$ and we are assuming $u^2 - 2 > 0$.

Having chosen n so that $\left(u - \frac{1}{n}\right)^2 > 2$ we claim now that $u - \frac{1}{n}$ is an upper bound for S. If not there $x \in S$ with $x > u - \frac{1}{n}$. Since $u \geq 1 \in S$ and $1/n \leq 1$ we have $u - \frac{1}{n} \geq 0$. So

$$x^2 > x\left(u - \frac{1}{n}\right) > \left(u - \frac{1}{n}\right)^2 > 2$$

contradicting $x \in S$.

So now we have shown that there is an upper bound $u - 1/n$ for S strictly smaller than the supposed least upper bound u. Thus (ii) is not possible.

Having eliminated (i) and (ii) we see that $u \notin \mathbb{Q}$, as claimed.

3. **(a)** Give the ε-N definition of the limit of a sequence of real numbers. Show that a bounded monotone sequence in $\mathbb{R}$ has a limit in $\mathbb{R}$.

   Solution: Definition 2.5 in the notes and proof of Theorem 2.14.

(b) Give an example of a bounded sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ which is not convergent. [Prove carefully that it is not convergent.]

Solution: The sequence $(x_n)_{n=1}^{\infty}$ where $x_n = (-1)^n$ is bounded since $-1 \leq x_n \leq 1$ for all n and so it is bounded above by 1 and below by -1. (Bounded means bounded above and bounded below.) It has no limit.

See Examples 2.10 part (iii) in the notes for one proof that it has no limit.

Another proof that the sequence $(x_n)_{n=1}^{\infty}$ where $x_n = (-1)^n$ has no limit is to use the fact that if $\lim_{n\to\infty} x_n = \ell$ then every subsequence $x_{n_j})_{j=1}^{\infty}$ has the same limit ℓ. In this example taking $n_j = 2j$ gives the constant sequence $x_{n_j} = x_{2j} = (-1)^{2j} = 1$ with limit 1. On the other hand taking $n_j = 2j+1$ gives the constant sequence $x_{n_j} = x_{2j+1} = (-1)^{2j+1} = -1$ with a different limit -1. So $\lim_{n\to\infty} x_n$ cannot exist.

(c) Show that every convergent sequence in $\mathbb{R}$ is bounded.

Solution: Suppose $(x_n)_{n=1}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} x_n = \ell \in \mathbb{R}$.

Then for $\varepsilon = 1 > 0$ we can find (by the definition of limit of a sequence) $N \in \mathbb{N}$ so that

$$n \geq N \Rightarrow |x_n - \ell| < 1.$$

Consequently

$$n \geq N \Rightarrow \ell - 1 \leq x_n \leq \ell + 1.$$

Take

$$L = \min(x_1, x_2, \ldots, x_{N-1}, \ell - 1)$$

(or $L = \ell - 1$ if it happens that $N = 1$) and

$$U = \max((x_1, x_2, \ldots, x_{N-1}, \ell + 1)$$

(or $L = \ell + 1$ if it happens that $N = 1$) and then we can say that

$$\text{for all } n \in \mathbb{N}, L \leq x_n \leq U$$

So the set $\{x_n : n \in \mathbb{N}\}$ is bounded below (by L) and above (by U). Thus the sequence is bounded.

[Note: In the proof of Theorem 2.9 (iii), this argument (or one similar) occurred.]

4. **(a)** Under what circumstances do we define $\lim_{x\to a} f(x)$ for a function $f(x)$ and $a \in \mathbb{R}$? How is the limit then defined?

Solution: (Definition 3.5 in the notes.)

Circumstances: $S \subset \mathbb{R}$ a subset, $f \colon S \to \mathbb{R}$ a real-valued function on S, $a \in S$ and S contains a punctured open interval about a.

Let $\ell \in \mathbb{R}$ be a number. Then we say that ℓ is a limit of f as x approaches a and write

$$\lim_{x\to a} f(x) = \ell$$

if the following holds:

for each sequence $(x_n)_{n=1}^{\infty}$ in $S \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$ it is true that $\lim_{n\to\infty} f(x_n) = \ell$.

Give the ε-δ criterion for $\lim_{x\to a} f(x) = \ell$ to be true. [That is, state the criterion but you are not asked to prove that it is a valid criterion.]

Solution: Theorem 3.9 in the notes.

For each $\varepsilon > 0$ it is possible to find $\delta > 0$ so that

$$|f(x) - \ell| < \varepsilon \text{ for each } x \in \mathbb{R} \text{ with } 0 < |x - a| < \delta.$$

Define continuity of a function $f\colon S \to \mathbb{R}$ at a point $a \in S$ (where $S \subset \mathbb{R}$).

Solution: Definition 3.10

(b) Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is continuous at every $a \neq 0$ but is not continuous at 0.

Solution: For $a \neq 0$ and any $\varepsilon > 0$, take $\delta = |a|$. Then if $a > 0$ we have $\delta = a$ and

$$|x - a| < \delta \Rightarrow 0 < x < 2a \Rightarrow f(x) = f(a) = 1 \Rightarrow |f(x) - f(a)| = 0 < \varepsilon$$

If $a < 0$, then $\delta = -a$ and

$$|x - a| < \delta \Rightarrow 2a < x < 0 \Rightarrow f(x) = f(a) = -1 \Rightarrow |f(x) - f(a)| = 0 < \varepsilon$$

Hence f is continuous at a (if $a \neq 0$).

If $a = 0$ we show that for $\varepsilon = 1/2$ it is impossible to find $\delta > 0$ so that

$$|x - a| = |x| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

No matter what $\delta > 0$ is chosen $x = \delta/2$ satisfies $|x| < \delta$ but $|f(x) - f(a)| = |1 - 0| = 1 \not< \varepsilon = 1/2$.

Show that $g\colon \mathbb{R} \to \mathbb{R}$ given by the rule $g(x) = (x^2 + x)f(x)$ is continuous at every $a \in \mathbb{R}$.

Solution: For $a \neq 0$ we have $\lim_{x\to a} f(x) = f(a)$ by continuity of f at a and $\lim_{x\to a}(x^2 + x) = a^2 + a$ by continuity of polynomials. So $\lim_{x\to a} g(x) = g(a)$ by the theorem on limits of products.

For $a = 0$ we could consider any sequence $(x_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = 0$. Then $\lim_{n\to\infty}(x_n^2 + x_n) = 0$ by continuity of the polynomial $x^2 + x$ at 0. The sequence $(f(x_n))_{n=1}^{\infty}$ is bounded (above by 1 and below by -1) and so

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} (x_n^2 + x_n) f(x_n) = 0$$

by one of the exercises.

Another way would be to take any $\varepsilon > 0$ given and to note that

$$|g(x) - g(0)| = |(x^2 + x)f(x)| \leq |x|^2 + |x|$$

by the triangle inequality and the fact that $|f(x)| \leq 1$ always. So if $|x| < 1$ we have

$$|g(x) - g(0)| \leq |x| + |x| = 2|x|.$$

Thus if we take $\delta = \min(1, \varepsilon/2)$ we have

$$|x - 0| < \delta \Rightarrow |x| < 1 \text{ and } |x| < \frac{\varepsilon}{2} \Rightarrow |g(x) - g(0)| \leq 2|x| < \varepsilon.$$

This shows continuity at 0 of g.