Chapter 3: Limits of functions (121 2004–05)

Remarks 3.1. *We move on now to start to deal with functions and most of the remainder of the course will be about functions. We will first discuss limits of functions* $\lim_{x\to a} f(x)$*, keeping in the back of our minds that we will define derivatives later via* $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

We did already define functions in 2.1 and more terminology about functions (for example: injective [or 'one-one'], surjective [or onto], bijective, inverse function) is part of course 111. Functions are used in essentially all parts of mathematics and in course 111 the emphasis is more on algebraic contexts (groups and mappings between them, for example, or permutations) but we will deal mostly with functions between sets of numbers and these can be visualised effectively in a graphical way.

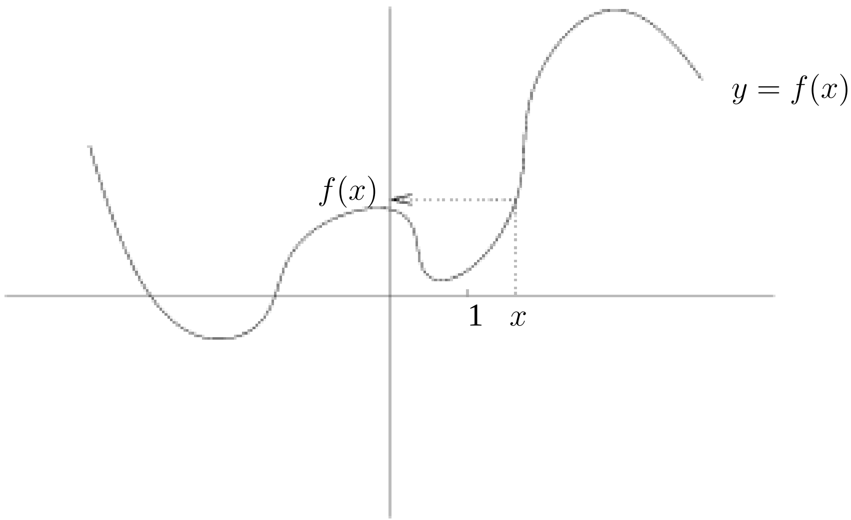
There is a Venn diagram approach to functions, treating sets as rather abstract blobs with elements indicated or marked as 'points' inside. Functions $f\colon A \to B$ can be thought of schematically as indicated by arrows starting at points a in the domain set A and ending at points $b = f(a) \in B$. There has to be exactly one arrow starting at each $a \in A$ (in this picture of f). Graphs are a more satisfactory picture for functions where A and B are the real numbers $\mathbb{R}$ or subsets of $\mathbb{R}$.

The *graph* of a function f is a set of ordered pairs

$$\text{Graph}(f) = \{(a, b) \in A \times B : b = f(a)\}$$

in the cartesian product of A and B (a subset G with the property that for each $a \in A$ there is a $b \in B$ so that $(a, b) \in G$ but also that if $(a, b_1) \in G$ and $(a, b_2) \in G$ then $b_1 = b_2$). When $A, B \subset \mathbb{R}$, we can picture the graph $\text{Graph}(f) \subset A \times B \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ as a set of points in the plane.

Typically (or in simple cases) this graph is a 'curve' in the plane with the property that each vertical line crosses it at most once.



The (vertical) line $x = x_0$ crosses the graph (once) if $x_0 \in A =$ the domain of f. A horizontal line $y = y_0$ crosses the graph if $y_0 = f(x)$ for some $x \in A$, that is if y_0 is in the *range* of f (which is $\{f(x) : x \in A\}$). The function is surjective (or onto) if each horizontal line $y = y_0$ with $y_0 \in B$ does cross the graph at least once. The function is injective exactly when each horizontal line crosses the graph at most once. [Explanation: If the line $y = y_0$ crosses the graph more than once it means that there are at least two $x_1, x_2 \in A$ with $f(x_1) = f(x_2) = y_0$ (and $x_1 \neq x_2$).]

Intervals 3.2. *In many cases we will be dealing with functions $f\colon A \to B$ where the sets A and B are* intervals*. Here we will review the notation for intervals.*

If $a, b \in \mathbb{R}$ and $a \leq b$, then the *closed interval* with end points a and b is the set of all real numbers between a and b inclusive of the end points:

$$[a, b] = \{x \in \mathbb{R} : a \leq x \textbf{ and } x \leq b\} = \{x \in \mathbb{R} : a \leq x \leq b\}$$

The *open interval* between a and b is

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

There are various infinite intervals (no endpoint on one side or the other) and we use the notations ∞ and $-\infty$ as convenient replacements for the missing end-points to the right or left. We do not mean to imply that there are any numbers ∞ or $-\infty$.

Here are the semi-infinite open and closed intervals ($a, b \in \mathbb{R}$) and the notations we use

$$\begin{aligned}
[a, \infty) &= \{x \in \mathbb{R} : a \leq x\} \\
(a, \infty) &= \{x \in \mathbb{R} : a < x\} \\
(-\infty, b] &= \{x \in \mathbb{R} : x \leq b\} \\
(-\infty, b) &= \{x \in \mathbb{R} : x < b\}
\end{aligned}$$

Note the convention that round brackets (or parentheses) are used for end points that are not included in the set. If all (finite) end points are included we refer to the interval as closed. If no finite intervals are included we call the interval open. So $[a, \infty)$ and $(-\infty, b]$ are closed, while (a, ∞) and $(-\infty, b)$ are open. There is one doubly infinite interval

$$(-\infty, \infty) = \mathbb{R}$$

and it counts as both open and closed.

There remain two other types of intervals we may encounter once or twice, the half-open and half-closed intervals (which are neither open nor closed)

$$\begin{aligned}
[a, b) &= \{x \in \mathbb{R} : a \leq x < b\} \\
(a, b] &= \{x \in \mathbb{R} : a < x \leq b\}
\end{aligned}$$

where we restrict to $a < b$.

Technically we could allow $a = b$ in the case $[a, b]$ but $[a, a] = \{a\}$ is just a one-point set and this will either be a very simple case or a case we will not want to consider in future theorems. The case (a, a) is the empty set and we will probably never want to consider that.

Examples 3.3. *For the function $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$, the range (set of values) turns out to be $\{y \in \mathbb{R} : y \geq 0\} = [0, \infty)$. We will prove this properly later. We may sometimes wish to discuses the same function x^2 but concentrate on a range of values of x like $0 \leq x \leq 1$ and then we are dealing with a different function $g: [0, 1] \to \mathbb{R}$ given by the same formula $g(x) = x^2$.*

At times we may want to have a surjective version of (essentially the same) function. Say $h: [0, 1] \to [0, 1]$ given by $h(x) = x^2$. Technically h and g are different because they have different co-domains but we may have to switch at times from g to h (and it is not such a huge distinction because the two functions g and h have identical domains and identical values).

Another type of example is a function given by a rule like $f(x) = 1/x$ which clearly does not make sense for $x = 0$ (when we would be trying to divide by 0). So the natural thing is to just make that restriction and consider $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = 1/x$. Here we have strayed outside having the domain A and co-domain B being intervals. $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ is a union of two intervals.

The set difference notation $A \setminus B$ means to take away from A any elements of B which may be in A. So $A \setminus B = \{x \in A : x \notin B\}$. For instance $[0, 2] \setminus [1, 3] = [0, 1)$ and $[0, 1] \setminus [3, 4] = [0, 1]$. (It makes no difference to A to take away all elements of B if none of the elements of B were in A to start with.)

Notation 3.4. *If $a \in \mathbb{R}$, then a punctured* open interval about a *means a subset of $\mathbb{R}$ of the form $(c, d) \setminus \{a\}$ where $c < a < b$.*

We can write $(c, d) \setminus \{a\} = (c, a) \cup (a, d)$ as a union of two open intervals on either side of a.

Definition 3.5. *Suppose $S \subset \mathbb{R}$ is a subset, $f\colon S \to \mathbb{R}$ is a real-valued function on S, $a \in S$ and suppose that S contains a punctured open interval about a. Let $\ell \in \mathbb{R}$ be a number. Then we say that ℓ is a limit of f as x approaches a and write*

$$\lim_{x \to a} f(x) = \ell$$

if the following holds:

> *for each sequence $(x_n)_{n=1}^\infty$ in $S \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$ it is true that $\lim_{n\to\infty} f(x_n) = \ell$.*

Remark. Limits are unique (if they exist).

Proof. We already know that a sequence can only have one limit (Proposition 2.7). Therefore as long as there is at least one sequence $(x_n)_{n=1}^\infty$ in $S \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$, we cannot have two values for $\lim x \to a f(x)$, because then $\lim_{n\to\infty} f(x_n)$ would have two values.

It is here that we rely on the fact that the domain S contains a punctured open interval about a. Say $b < a < c$ and $(b, c) \setminus \{a\} \subset S$. Then $x_n = a + (c - a)/(n + 1)$ is a valid choice of a sequence $(x_n)_{n=1}^\infty$ in $S \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$. □

Examples 3.6. (i) $\lim_{x\to 2} x^2 = 4$

Proof. Note that $f(x) = x^2$ makes sense for all $x \in \mathbb{R}$ and so we can treat f as a function $f\colon \mathbb{R} \to \mathbb{R}$. Then the domain $\mathbb{R}$ certainly contains a punctured open interval about 2 and we are in a position to contemplate $\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2$.

[**Aside.** The fact that $f(2)$ actually makes sense is not relevant and we will never use that in the course of the proof. While calculating $\lim_{x\to 2} x^2$ we never use $x = 2$ at all. (The reason for this is that the kind of limits we will encounter later when dealing with derivatives are of the form $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ and the faction there does not make sense if we had $h = 0$.)]

Now take any sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R} \setminus \{2\}$ with $\lim_{n\to\infty} x_n = 2$. Then we have

$$\lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} x_n x_n = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} x_n\right)$$

by the theorem on limits of products (of sequences). Thus we get $2 \times 2 = 4$.

Since $\lim_{n\to\infty} x_n^2 = 4$ no matter which sequence $(x_n)_{n=1}^\infty$ we take in $\mathbb{R} \setminus \{2\}$ with $\lim_{n\to\infty} x_n = 2$, we have shown that $\lim_{x\to 2} x^2 = 4$. $\square$

The next two examples will be building blocks for future use.

(ii) For any $a \in \mathbb{R}$, $\lim_{x\to a} x = a$.

Proof. This is really easy to show, because the criterion is self-evidently true.

We can consider the function to be $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ and again there is no question but that the domain contains a punctured open interval about a. (We could write down $(a-1, a+1) \setminus \{a\}$ if we want to see a specific punctured open interval, but there are many possible choices.) We start with any sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R} \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$. Then we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = a$$

automatically true.

So we have shown $\lim_{x\to a} x = a$. $\square$

(iii) For any $a \in \mathbb{R}$, and any (constant) $\lambda \in \mathbb{R}$, $\lim_{x\to a} \lambda = \lambda$.

Proof. In this case the function is the constant one $f(x) = \lambda$. Start with any sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R} \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$. with $\lim_{n\to\infty} x_n = a$. Then we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \lambda = \lambda \lim_{n\to\infty} 1 = \lambda \times 1 = \lambda.$$

□

Theorem 3.7. *Suppose f and g are are $\mathbb{R}$-valued functions defined on subsets of $\mathbb{R}$. Suppose also $a, \ell, m \in \mathbb{R}$, $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$. Then*

(i) $\lim_{x\to a}(f(x) + g(x)) = \ell + m$

(ii) $\lim_{x\to a} f(x)g(x) = \ell m$

(iii) *if* $m \neq 0$, $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\ell}{m}$

Proof. (i) The first thing to settle is that the limit of the sum makes sense. We are assuming that $f\colon S_1 \to \mathbb{R}$, $g\colon S_2 \to \mathbb{R}$ where $S_1, S_2 \subset \mathbb{R}$ are subsets of $\mathbb{R}$ that each contain a punctured open interval about a (so that the limits we are assuming to exist can fit into the conditions for Definition 3.5 above).

We can reasonably define $f + g$ as the function $f + g\colon S_1 \cap S_2 \to \mathbb{R}$ by the rule $(f + g)(x) = f(x) + g(x)$ and to consider the limit of the sum we should know that $S_1 \cap S_2$ contains a punctured open interval about a. Say c_1, d_1, c_2, d_2 are chosen so that $a \in (c_1, d_1)$, $(c_1, d_1) \setminus \{a\} \subset S_1$, $a \in (c_2, d_2)$, and $(c_2, d_2) \setminus \{a\} \subset S_2$. Then $a \in (c_1, d_1) \cap (c_2, d_2) = (c, d)$ where $c = \max(c_1, c_2)$, $d = \min(d_1, d_2)$ and $(c, d) \setminus \{a\} \subset S_1 \cap S_2$. So the conditions are right to consider $\lim_{x\to a}(f(x) + g(x))$.

Take any sequence $(x_n)_{n=1}^\infty$ in $(S_1 \cap S_2) \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$. Then $\lim_{n\to\infty} f(x_n) = \ell$ and $\lim_{n\to\infty} g(x_n) = m$. It follows from Theorem 2.9 that $\lim_{n\to\infty} f(x_n) + g(x_n) = \ell + m$. As this is the case for all possible sequences $(x_n)_{n=1}^\infty$ as above, we have shown (i).

(ii) Essentially the same proof works for products as for sums. Again the product function fg can be defined on $S_1 \cap S_2$ by $(fg)(x) = f(x)g(x)$. The conclusion $\lim_{n\to\infty} f(x_n)g(x_n) = \ell m$ follows by Theorem 2.9 (for any sequence $(x_n)_{n=1}^\infty$ in $(S_1 \cap S_2) \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$). Thus (ii) follows.

(iii) Here there is a more complicated issue. We can define the quotient $f(x)/g(x)$ when we are not dividing by 0 (and when both $f(x)$ and $g(x)$ make sense). Thus it makes sense for x in the set

$$S = \{x \in S_1 \cap S_2 : g(x) \neq 0\}$$

and before we can discuss $\lim_{x\to a} f(x)/g(x)$ we should know that S contains a punctured open interval about a. This requires proof.

We know that $S_1 \cap S_2$ contains a punctured open interval $(c, d) \setminus \{a\}$ (some $c < a < d$) about a. It will be more convenient for this argument to have a symmetric punctured open interval (with a in its centre). For this we take $\delta = \min(a - c, d - a)$. Then $\delta > 0$ and $(a - \delta, a + \delta) \subset (c, d)$. So we have $(a - \delta, a + \delta) \setminus \{a\} \subset S_1 \cap S_2$.

We *claim* that S contains a punctured open interval about a.

It could be that $g(x)$ is never zero in $(a - \delta, a + \delta) \setminus \{a\}$ and if that is the case we have a punctured open interval about a contained in S and the claim is true. If not, there is at least one $x_1 \in (a - \delta, a + \delta) \setminus \{a\}$ with $g(x_1) = 0$. Fix one such x_1.

Now consider $(a - \delta/2, a + \delta/2) \setminus \{a\}$. Either $g(x)$ is never zero on that punctured open interval (and so the claim is true) or there is $x_2 \in (a - \delta/2, a + \delta/2) \setminus \{a\}$ with $g(x_2) = 0$.

In general, for $n = 3, 4, \ldots$, either $g(x)$ is never 0 on the punctured open interval $(a - \delta/n, a + \delta/n) \setminus \{a\}$ (and then the claim is true) or there is $x_n \in (a - \delta/n, a + \delta/n) \setminus \{a\}$ with $g(x_n) = 0$.

If we never find n with $(a - \delta/n, a + \delta/n) \setminus \{a\} \subset S$, then we can find an infinite sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in (a - \delta/n, a + \delta/n) \setminus \{a\}$ with $g(x_n) = 0$. Since $|x_n - a| < \delta/n$ it follows fairly easily that $\lim_{n\to\infty} x_n = a$. Since $\lim_{x\to a} g(x) = m$ is assumed to be the case and $x_n \in S \setminus \{a\}$, we have $\lim_{n\to\infty} g(x_n) = m$. But, as $g(x_n) = 0$ for each n this means $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} 0 = 0$ and this leads to $0 = m$, contradiction the assumption $m \neq 0$. This contradiction leads to the conclusion that the claim must hold.

Take any sequence $(x_n)_{n=1}^{\infty}$ in $S \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$.
Then $\lim_{n\to\infty} f(x_n) = \ell$ and $\lim_{n\to\infty} g(x_n) = m$. It follows from Theorem 2.9 that $\lim_{n\to\infty} f(x_n)/g(x_n) = \ell/m$. As this is the case for all possible sequences $(x_n)_{n=1}^{\infty}$ as above, we have shown (iii).

$\square$

Examples 3.8. (i) Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{j=0}^n a_j x^j$ be a *polynomial* function. (If $a_n \neq 0$ so that the x^n term is actually present in the sum, then the polynomial is said to be of *degree* n.)

We can prove by induction on n that $\lim_{x\to a} x^n = a^n$ for $n = 1, 2, 3, \ldots$ and any $a \in \mathbb{R}$ (and even for $n = 0$ if we interpret x^0 as standing for the constant function 1). We have already verified the case of the constant and the case $n = 1$. Once we have checked $\lim_{x\to a} x^k = a^k$ for a particular $k \in \mathbb{N}$ we can see that

$$\lim_{x\to a} x^{k+1} = \lim_{x\to a} x^k x = \lim_{x\to a} x^k \lim_{x\to a} x = a^k a = a^{k+1}$$

(using Theorem 3.7 (ii)). By induction $\lim_{x\to a} x^n = a^n$ for all $n \in \mathbb{N}$ (and $a \in \mathbb{R}$).

We can then also prove by induction on the degree n of the polynomial $p(x)$ that $\lim_{x\to a} p(x) = p(a)$. For the case $n = 1$ we have

$$\lim_{x\to a} a_0 + a_1 x = \lim_{x\to a} a_0 + \lim_{x\to a} a_1 x = a_0 + (\lim_{x\to a} a_1)(\lim_{x\to a} x) = a_0 + a_1 a$$

For the inductive step, if we know that

$$\lim_{x\to a} \sum_{j=0}^k a_j x^j = \sum_{j=0}^k a_j a^j$$

for all $a, a_0, a_1, \ldots, a_k$, then

$$\begin{aligned}
\lim_{x\to a} \sum_{j=0}^{k+1} a_j x^j &= \lim_{x\to a} \left(\sum_{j=0}^k a_j x^j \right) + a_{k+1} x^{k+1} \\
&= \lim_{x\to a} \left(\sum_{j=0}^k a_j x^j \right) + \lim_{x\to a} a_{k+1} x^{k+1} \\
&= \sum_{j=0}^k a_j a^j + \left(\lim_{x\to a} a_{k+1} \right) \left(\lim_{x\to a} x^{k+1} \right) \\
&= \sum_{j=0}^k a_j a^j + a_{k+1} a^{k+1} \\
&= \sum_{j=0}^{k+1} a_j a^j
\end{aligned}$$

(ii) A rational function is a function of the form $r(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials and $q(x)$ is not the zero polynomial. The domain of $r(x)$ is the set $S = \{x \in \mathbb{R} : q(x) \neq 0\}$.

From the previous facts about limits of polynomials we can show that if $a \in S$, then

$$\lim_{x \to a} r(x) = r(a)$$

by using Theorem 3.7 (iii).

Theorem 3.9. *Suppose $f\colon S \to \mathbb{R}$ is a function defined on a subset $S \subset \mathbb{R}$ that contains a punctured open interval about some point $a \in \mathbb{R}$. Let $\ell \in \mathbb{R}$. Then $\lim_{x\to a} f(x) = \ell$ holds if and only if the following ε-δ* criterion *is satisfied:*

For each $\varepsilon > 0$ it is possible to find $\delta > 0$ so that

$$|f(x) - \ell| < \varepsilon \text{ for each } x \in \mathbb{R} \text{ with } 0 < |x - a| < \delta.$$

Proof. A statement of this "if and only if" type contains two assertions in one and both have to be proved independently. There is an 'implies' or 'only if' direction $\Rightarrow$ which is that if the first statement $\lim_{x\to a} f(x) = \ell$ is true then the second statement (the ε-δ criterion) must hold. In addition there is the 'if' or reverse implication direction $\Leftarrow$ where we must show that if the second statement (the ε-δ criterion) is valid then $\lim_{x\to a} f(x) = \ell$ must hold.

The net effect is to show that the two statements are equivalent in the sense that any time either one of them is valid, then the other is also valid. Of course, if any one of them is not valid, then the other is also false.

$\Rightarrow$: Assume now we know $\lim_{x\to a} f(x) = \ell$ is true. Then the domain S of f must contain a punctured open interval about a and as in the early part of the proof of Theorem 3.7 (iii) above we can assume that there is a symmetric punctured open interval $(a - \delta_0, a + \delta_0) \setminus \{a\} \subset S$ for some $\delta_0 > 0$. (We use δ_0 now because we will need a different δ in a moment.)

To establish the ε-δ criterion, start with $\varepsilon > 0$ given. We claim there must be a suitable $\delta > 0$ so that

$$|f(x) - \ell| < \varepsilon \text{ for each } x \in \mathbb{R} \text{ with } 0 < |x - a| < \delta.$$

If $\delta = \delta_0/n$ does not work for any $n \in \mathbb{N}$, then (since all x with $0 < |x - a| < \delta_0/n$ have $0 < |x - a| < \delta_0$, and so all such x are in $(a - \delta_0, a + \delta_0) \setminus \{a\} \subset S$) there must be x_n with $0 < |x_n - a| < \delta_0/n$ and $|f(x_n) - \ell| \geq \varepsilon$. Now $(x_n)_{n=1}^{\infty}$

would be a sequence in $S \setminus \{a\}$, $\lim_{n\to\infty} x_n = a$ (as is easy to check based on $|x_n - a| < \delta_0/n$) and yet $\lim_{n\to\infty} f(x_n) \neq \ell$. This contradicts the assumption that $\lim_{x\to a} f(x) = \ell$.

The contradiction arose by assuming that the claim is not satisfied by $\delta = \delta_0/n$ for any $n \in \mathbb{N}$. So there is a $\delta > 0$ that satisfies the claim.

$\Leftarrow$: Suppose now that the ε-δ criterion holds (for a, f and ℓ). To show that $\lim_{x\to a} f(x) = \ell$, consider any sequence $(x_n)_{n=1}^{\infty}$ in $S \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$. To show that $\lim_{n\to\infty} f(x_n) = \ell$ (according to the ε-N definition of limits of sequences) let $\varepsilon > 0$ be given. Then we can find $\delta > 0$ according to the criterion we are assuming to be valid so that (for these particular ε and δ)

$$|f(x) - \ell| < \varepsilon \text{ for each } x \in \mathbb{R} \text{ with } 0 < |x - a| < \delta.$$

Now using the ε-N definition of limits of sequences with 'ε'$= \delta$ we know we can find $N \in \mathbb{N}$ so that

$$|x_n - a| < \delta \text{ for all } n \geq N.$$

Putting these two statements together with the fact that $x_n \neq a$ for all n, we have

$$n \geq N \Rightarrow 0 < |x_n - a| < \delta \Rightarrow |f(x_n) - \ell| < \varepsilon.$$

The fact that we can find such an N for any given $\varepsilon > 0$ establishes $\lim_{n\to\infty} f(x_n) = \ell$. As this is true for all sequences $(x_n)_{n=1}^{\infty}$ in $S \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$ we have shown $\lim_{x\to a} f(x) = \ell$. $\square$

Definition 3.10. *If $f\colon S \to \mathbb{R}$ is a function on a subset $S \subset \mathbb{R}$ and $a \in S$ then f is called* continuous at a *if the following ε-δ* criterion *is satisfied:*

> *For each $\varepsilon > 0$ it is possible to find $\delta > 0$ so that*
>
> $$x \in S, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Example 3.11. *Polynomial functions $p(x) = \sum_{j=0}^{n} a_j x^j$ are continuous at every $a \in \mathbb{R}$.*

Proof. Here $p\colon \mathbb{R} \to \mathbb{R}$. We already know that $\lim_{x\to a} p(x) = p(a)$ (examples above) and we use the fact that this can be restated via an ε-δ criterion (Theorem 3.9).

Starting with $\varepsilon > 0$ given Theorem 3.9 tells us we can find $\delta > 0$ so that

$$0 < |x - a| < \delta \Rightarrow |p(x) - p(a)| < \varepsilon.$$

But $x = a$ gives $|p(x) - p(a)| = |p(a) - p(a)| = 0 < \varepsilon$ and so we do not need to rule out $|x - a| = 0$ in this situation. We have then

$$|x - a| < \delta \Rightarrow |p(x) - p(a)| < \varepsilon$$

and this means we have found $\delta > 0$ to ensure the criterion of Definition 3.10 holds for the given $\varepsilon > 0$. As we can do this for any $\varepsilon > 0$, we have established continuity of $p(x)$ at a. $\square$

Definition 3.12. *If $S \subset \mathbb{R}$ is a subset of $\mathbb{R}$ and $a \in S$, then a is called an* interior point of S *if there is an open interval that contains a and is contained in S.*

In other words, if there are $c < d$ with $a \in (c, d) \subset S$.

Proposition 3.13. *Let $f \colon S \to \mathbb{R}$ be a function defined on $S \subset \mathbb{R}$ and $a \in S$ an interior point of S. Then f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$.*

Proof. With a little care, this follows from Theorem 3.9. There is a restriction to $x \in S$ in the definition of continuity which is not present in the ε-δ condition of Theorem 3.9, and in the theorem the restriction $0 < |x - a|$ is present to avoid considering $x = a$ while taking the limit. Since we are dealing with an interior point of S, the $x \in S$ condition can be subsumed in $|x - a| < \delta$ if we ensure that δ is reasonably small. Since the limit is $f(a)$ the condition $0 < |x - a|$ is not needed.

$\Rightarrow$: First choose $c < a < d$ with $(c, d) \subset S$. Put $\delta_0 = \min(a-c, d-a)$ and then we have $\delta_0 > 0$ with $(a-\delta_0, a+\delta_0) \subset S$. Our aim is to show $\lim_{x\to a} f(x) = f(a)$ via the ε-δ condition of Theorem 3.9.

Let $\varepsilon > 0$ be given. Applying the definition of continuity at a we can find $\delta' > 0$ so that

$$x \in S, |x - a| < \delta' \Rightarrow |f(x) - f(a)| < \varepsilon.$$

We use δ' because we now set $\delta = \min(\delta_0, \delta')$. Then

$$|x-a| < \delta \Rightarrow |x-a| < \delta_0 \text{ and } |x-a| < \delta' \Rightarrow x \in S \text{ and } |x-a| < \delta' \Rightarrow |f(x)-f(a)| < \varepsilon.$$

By Theorem 3.9 $\lim_{x\to a} f(x) = f(a)$.

$\Leftarrow$: Assume now that $\lim_{x\to a} f(x) = f(a)$. To show continuity at a, let $\varepsilon > 0$ be given. By Theorem 3.9 we can find $\delta > 0$ so that

$$0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

The restriction $0 < |x-a|$ is not needed since $|f(x) - f(a)| < \varepsilon$ is certainly true for $x = a$. Thus

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

It follows that

$$x \in S, |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon,$$

which shows that the δ we got from the theorem satisfies the condition in the definition of continuity.

Since we can find such a $\delta > 0$ for each initial $\varepsilon > 0$, we have shown that f must be continuous at a. $\square$

Definition 3.14. *If $S \subset \mathbb{R}$ and $a \in S$, then a is called an* isolated point of S *if there is an open interval around a that contains no point of S apart from a.*

That is, if there exist $c < a < d$ so that $(c,d) \cap S = \{a\}$.

Lemma 3.15. *If $S \subset \mathbb{R}$, $f \colon S \to \mathbb{R}$ and $a \in S$ is an isolated point of S, then f is automatically continuous at a.*

Proof. Choose $c < a < d$ with $(c,d) \cap S = \{a\}$. Put $\delta = \min(a-c, d-a)$ and then

$$x \in S, |x-a| < \delta \Rightarrow x \in (a-\delta, a+\delta) \cap S \subset c,d) \cap S = \{a\} \Rightarrow x = a \Rightarrow f(x) - f(a) = 0.$$

Thus for any $\varepsilon > 0$ and this choice of $\delta > 0$ we have

$$x \in S, |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

$\square$

Example 3.16. *Let $S = (0,1) \cup \{2\}$. Then 2 is an isolated point of S and every other point is an interior point. If $f \colon S \to \mathbb{R}$, then f is automatically continuous at 2 and continuity at $a \in (0,1)$ is equivalent to $\lim_{x \to a} f(x) = f(a)$.*

At least at interior points, we can view the continuity condition as a stability condition. A small change in the value of x away from $x = a$ produces a small change in the value $f(x)$ away from $f(a)$.

The ε-δ definition of continuity makes this more precise. ε is to be interpreted as a precise meaning for $f(x)$ to be close to $f(a)$ and the idea is that, having fixed that, there is a way to interpret x close the a (the distance δ) so that when x is close to a in this sense then $f(x)$ is 'close' to $f(a)$ in the desired sense.

For practical purposes where it is common to compute with decimal approximations, this is important because it says that if you do the calculations sufficiently accurately (though still not exactly) you will get an accurate value for $f(a)$. If the function is discontinuous at a, the smallest approximation made in $x \cong a$ could produce a big change in $f(x)$.

For points that are not interior points, the restriction that $x \in S$ can be at least as important as the one that x is 'close' to a. In the extreme case of an isolated $a \in S$ we see that continuity at a does not place any condition on the function.

Remark 3.17. *We introduced limits of functions in Definition 3.5 by relying on limits of sequences and we proved in Theorem 3.9 that an alternative approach via an ε-δ criterion would yield the same concept. For continuity (at a point) we used a definition based on ε-δ in 3.10. Now we show that a sequence approach also works for continuity.*

Theorem 3.18. *Let $S \subset \mathbb{R}$, $f\colon S \to \mathbb{R}$ a function ad $a \in S$ a point. Then f is continuous at a if and only if the following holds*

> *for each sequence $(x_n)_{n=1}^\infty$ of terms $x_n \in S$ with $\lim_{n\to\infty} x_n = a$ we have $\lim_{n\to\infty} f(x_n) = f(a)$.*

Proof. $\Rightarrow$: Assume now that f is continuous and let $(x_n)_{n=1}^\infty$ be a sequence in S converging to a. To show $\lim_{n\to\infty} f(x_n) = f(a)$ by using the definition of limit of a sequence directly, we take $\varepsilon > 0$ given and we claim there is $N \in \mathbb{N}$ so that

$$n \geq N \Rightarrow |f(x_n) - f(a)| < \varepsilon.$$

By continuity we know we can find $\delta > 0$ so that

$$x \in S, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Using the ε-N definition of what it means for $\lim_{n\to\infty} x_n = a$ with the role of positive number 'ε' taken by $\delta > 0$ we deduce that there is $N \in \mathbb{N}$ so that

$$n \geq N \Rightarrow |x_n - a| < \delta.$$

Putting these two statements together we have

$$n \geq N \Rightarrow |x_n - a| < \delta \text{ and } x_n \in S \Rightarrow |f(x_n) - f(a)| < \varepsilon,$$

and so we have N as required.

As we can find N for each $\varepsilon > 0$ given, we have shown $\lim_{n\to\infty} f(x_n) = f(a)$.

$\Leftarrow$: Assume now that we have the information about sequences and we aim to show that f is continuous at a. Let $\varepsilon > 0$ be given and we claim there is some $\delta > 0$ so that

$$x \in S, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

If that is not the case, then $\delta = 1/n$ will not satisfy this and so there must be some $x_n \in S$ so that $|x_n - a| < 1/n$ but $|f(x_n) - f(a)| \geq \varepsilon$. Chose such an x_n for each $n = 1, 2, 3, \ldots$. It is easy to see then that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} f(x_n) \neq f(a)$ contradicting the assumption.

So $\delta = 1/n$ must satisfy the desired implication (for some $n \in \mathbb{N}$). $\square$

Definition 3.19. *If $S \subset \mathbb{R}$ and $f \colon S \to \mathbb{R}$ is a function, then f is called* continuous on S *if f is continuous at each point $a \in S$.*

Corollary 3.20. *Let $S \subset \mathbb{R}$ and let $f, g \colon S \to \mathbb{R}$ be two continuous functions.*

(i) The function $f + g \colon S \to \mathbb{R}$ defined by the rule $(f + g)(x) = f(x) + g(x)$ (for $x \in S$) is continuous.

In short 'sums of continuous functions are continuous'.

(ii) The function $fg \colon S \to \mathbb{R}$ defined by the rule $(fg)(x) = f(x)g(x)$ (for $x \in S$) is continuous.

In short 'products of continuous functions are continuous'.

(iii) If $g(x) \neq 0$ for each $x \in S$, then the function $\frac{f}{g} \colon S \to \mathbb{R}$ defined by the rule $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (for $x \in S$) is continuous.

In short 'quotients of continuous functions are continuous (as long as the denominator is never zero)'.

Proof. It is quite easy to use Theorem 3.18 to verify these claims.

(i) Let $a \in S$ and let $(x_n)_{n=1}^{\infty}$ a be a sequence in S that converges to a. Then $\lim_{n\to\infty} f(x_n) = f(a)$ and $\lim_{n\to\infty} g(x_n) = g(a)$ by continuity of f and g (using Theorem 3.18). By the theorem on limits of sums of sequences (2.9), we deduce $\lim_{n\to\infty} f(x_n) + g(x_n) = f(a) + g(a)$ and so (using Theorem 3.18 again) $f + g$ is continuous at a.

Since this is true for all $a \in S$, we have $f + g$ continuous (on S).

(ii)

(iii) The proofs are essentially the same.

$\square$

Notation 3.21. *If* $f: S \to \mathbb{R}$ *is a function and* $T \subset S$, *then the* restriction of f to T *is the function* $f\,|_T: T \to \mathbb{R}$ *given by the rule* $(f\,|_T)(x) = f(x)$ *for* $x \in T$.

We can think of the restriction as forgetting about some of the values $f(x)$ of the original functions f (and only retaining those where $x \in T$). In terms of the graphs, the graph of the restriction would be just part of the graph of f.

It is very easy to see from either the definition of continuity or Theorem 3.18 that the restriction of a continuous function will be continuous. More precisely, $a \in T \subset S$ and $f: S \to \mathbb{R}$ is continuous at a, then $f\,|_T$ is continuous at a. Starting with $\varepsilon > 0$ given, continuity of f at a says there is $\delta > 0$ so that

$$x \in S, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Thus

$$x \in T, |x - a| < \delta \Rightarrow x \in S, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

and so

$$x \in T, |x - a| < \delta \Rightarrow |(f\,|_T)(x) - (f\,|_T)(a)| < \varepsilon.$$

Definition 3.22. *If* $S, T \subset \mathbb{R}$, $f: S \to T$ *and* $g: T \to \mathbb{R}$, *then the* composition *of* f *and* g *(also known as* g after f*) is the function* $g \circ f: S \to \mathbb{R}$ *given by the rule* $(g \circ f)(x) = g(f(x))$ *for* $x \in S$.

We pronounce $g \circ f$ as 'g circle f'.

It is possible to define $g \circ f$ when the co-domain of f is not the domain of g, if we ask just that $f(S) \subset T$ (the range of f is contained in the domain of g), but there are various abstract settings where this in not the way to do things and so we will stick officially to the definition as given.

There is a technical difference between a function $f: S \to T$ where $T \subset \mathbb{R}$ and the functions $: S \to \mathbb{R}$ and $: S \to f(S)$ given by the same rule $x \mapsto f(x)$. For example surjectivity can depend on which function we consider (which co-domain we take). However, in many ways the differences are technical.

We do consider a function $f: S \to T$ to be continuous when the (almost same) function $: S \to \mathbb{R}$ ($x \mapsto f(x)$) is continuous.

Theorem 3.23. *Let* $S, T \subset \mathbb{R}$ *be subsets, and* $f: S \to T$ *and* $g: T \to \mathbb{R}$ *two continuous functions. Then the* composition *of* $g \circ f: S \to \mathbb{R}$ *is continuous.*

Proof. Fix $a \in S$ and then we will show that $g \circ f$ is continuous at a via the sequence criterion Theorem 3.18. Let $(x_n)_{n=1}^\infty$ be a sequence in S that converge to a (that is with $\lim_{n\to\infty} x_n = a$). Then $\lim_{n\to\infty} f(x_n) = f(a)$ by continuity of f at a.

Now for $y_n = f(x_n)$, $(y_n)_{n=1}^\infty$ is a sequence in T that converges to $b = f(a) \in T$. By continuity of g at b (and Theorem 3.18 again) we have $\lim_{n\to\infty} g(y_n) = g(b)$. That is

$$\lim_{n\to\infty} g(f(x_n)) = g(f(a)),$$

or

$$\lim_{n\to\infty} (g \circ f)(x_n) = (g \circ f)(a).$$

As this is true for all sequences $(x_n)_{n=1}^\infty$ in S converging to a, it implies that $g \circ f$ is continuous at a.

As this holds for each $a \in S$, it follows that $g \circ f$ is continuous. □

Remark 3.24. The idea of using different variables for different functions can help to keep track of compositions. If we write $y = f(x)$ and $z = g(y)$, then $z = g(y) = g(f(x))$ is almost automatic.

Alternatively, we may use $u = f(x)$ and $y = g(u)$ with $y = g(f(x))$.

Remark 3.25. We have 2 ways to characterise continuity of a function $f\colon S \to \mathbb{R}$ at any point $a \in S$ — the original ε-δ definition and the sequence criterion Theorem 3.18.

For interior points $a \in S$, we also have the criterion $\lim_{x\to a} f(x) = f(a)$ by Proposition 3.13.

We can also use a variation of a limit criterion at either end of a closed interval. If $f\colon [a, b] \to \mathbb{C}$ we can characterise continuity at a via a one-sided limit (and we can do something similar at b).

Definition 3.26. *1. Let $f\colon S \to \mathbb{R}$ be defined on a set S that includes and open interval (a, b) with $a < b$. Let $\ell \in \mathbb{R}$. Then we say that the* right hand limit $\lim_{x\to a^+} f(x)$ *is ℓ if the following ε-δ criterion holds:*

For each $\varepsilon > 0$ it is possible to find $\delta > 0$ so that

$$0 < x - a < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

2. Let $f\colon S \to \mathbb{R}$ be defined on a set S that includes and open interval (c, a) with $c < a$. Let $\ell \in \mathbb{R}$. Then we say that the left hand limit $\lim_{x\to a^-} f(x)$ *is ℓ if the following ε-δ criterion holds:*

> *For each $\varepsilon > 0$ it is possible to find $\delta > 0$ so that*
>
> $$0 < a - x < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

Proposition 3.27. *Let $f : [a, b] \to \mathbb{R}$ be a function on an interval where $a < b$.*

(i) f is continuous at a if and only if $\lim_{x \to a^+} f(x) = f(a)$.

(ii) f is continuous at b if and only if $\lim_{x \to b^-} f(x) = f(b)$.

Proof. (of (i) as (ii) is similar).

Continuity of f at a means that

> For each $\varepsilon > 0$ it is possible to find $\delta > 0$ so that
>
> $$x \in [a, b], |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

The criterion for $\lim_{x \to a^+} f(x) = f(a)$ has two differences. There is no $x \in [a, b]$ and $x = a$ is not considered. However the point about allowing $x = a$ is not a problem since when $x = a$ we automatically have $|f(x) - f(a)| = 0 < \varepsilon$.

Notice that

$$x \in [a, b], |x - a| < \delta \Rightarrow 0 \leq |x - a| < \delta$$

and if we assume $\delta < b - a$, we can say

$$0 \leq |x - a| < \delta \Rightarrow x \in [a, b], |x - a| < \delta$$

To make a formal proof we need to show 2 things.

f is continuous at $a \Rightarrow \lim_{x \to a^+} f(x) = f(a)$. :

Start with $\varepsilon > 0$. From continuity at a find $\delta > 0$ so that

$$x \in [a, b], |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Take $\delta_0 = \min(\delta, b - a)$ and then we have $\delta_0 > 0$ and

$$\begin{aligned} 0 < x - a < \delta_0 &\Rightarrow x \in [a, b], |x - a| < \delta \\ &\Rightarrow |f(x) - f(a)| < \varepsilon \end{aligned}$$

so that we have shown that the criterion for $\lim_{x \to a^+} f(x) = f(a)$ holds.

$\lim_{x\to a^+} f(x) = f(a) \Rightarrow f$ **is continuous at** a**.** :

Start with $\varepsilon > 0$. From $\lim_{x\to a^+} f(x) = f(a)$, we can find $\delta > 0$ so that

$$0 < x - a < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Now,

$$x \in [a, b], |x - a| < \delta \Rightarrow 0 \leq x - a < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

because it is true for $x = a$ as well as for $0 < x - a < \delta$. Thus f is continuous at a.

$\square$