Chapter 2: Sequences I (121 2004–05)

We will discuss (infinite) sequences (of numbers) and limits of them, but we will postpone several aspects to later. We will also leave series to later.

Definition 2.1. *A* function *with* domain *a set A and co-domain (or target) a set B is a rule that assigns one and only one element $b \in B$ to each element $a \in A$.*

We usually write $f: A \to B$ to indicate that we are discussing a function called f with domain A and co-domain B. We normally write $f(a)$ for the element $b \in B$ that is associated to $a \in A$ by the rule f.

We will have more to say about functions later, and they are also discussed in course 111.

Definition 2.2. *A* sequence *(an infinite sequence) of numbers is a function $x: \mathbb{N} \to \mathbb{R}$.*

In contrast to the usual notation $x(n)$ for function values, it is traditional to denote $x(n)$ by the subscript notation x_n when dealing with sequences. We will often write $(x_n)_{n=1}^\infty$ for a sequence and this is because we think of the values of the sequence as forming an infinite list

$$x_1, x_2, x_3, \ldots$$

Examples 2.3. *(i) $x_n = n^2$ gives the sequence $(x_n)_{n=1}^\infty = (n^2)_{n=1}^\infty$ which can be listed as*

$$1^2, 2^2, 3^2, 4^2, \ldots = 1, 4, 9, 16, 25, \ldots$$

(ii) $x_n = n^2$ gives the sequence $(x_n)_{n=1}^\infty = ((-1)^n)_{n=1}^\infty$ which can be listed as

$$-1, 1, -1, 1, -1, \ldots$$

(iii) $\left(\frac{1}{n}\right)_{n=1}^\infty$ is the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Remark 2.4. Recall that for $x, y \in \mathbb{R}$, we can interpret $|x - y|$ geometrically as the *distance* between x and y on the number line. Later there will be courses about distances in the abstract and then this will be the definition of one possible distance to use on $\mathbb{R}$.

For now we can think of it as the obvious definition of distance on the number line and use it as a guide to understanding what we are doing. It will not actually be an essential ingredient in any proof or definition that we give.

The idea of a limit of a sequence $(x_n)_{n=1}^\infty$ being a number $\ell \in \mathbb{R}$ is what we want to introduce now. We want to describe the notion that x_n is getting "closer and closer" to ℓ as n gets "larger and larger" in a way that is unambiguous and gives the right concept.

To make the notion precise we have to be exact about what 'close' means and what 'large' means. Our approach is to say that 'close' means distance less than some positive quantity. This is then combined with the idea the that positive quantity can be fixed to any positive value so as to accommodate different notions of 'small distance'. It is usual to use the notation ε (Greek letter epsilon) for this positive quantity.

We also have to have a notion of 'large' and for this we choose a integer N and say that any $n \in \mathbb{N}$ with $n \geq N$ will be considered large.

So x_n 'closer and closer' to ℓ will become $|x_n - \ell| < \varepsilon$ and n getting larger and larger will mean all $n \geq N$. It will be important though to understand the logic.

Definition 2.5. *If $(x_n)_{n=1}^\infty$ is a sequence of real numbers and $\ell \in \mathbb{R}$, then we say that ℓ is a* limit *of the sequence if the following is true:*

For each $\varepsilon > 0$ it is possible to find $N \in \mathbb{N}$ so that

$$|x_n - \ell| < \varepsilon \textit{ holds for every } n \in \mathbb{N} \textit{ with } n \geq N.$$

We call a sequence $(x_n)_{n=1}^\infty$ convergent *if it has a limit $\ell \in \mathbb{R}$.*

The logic is important to grasp. No matter how small (or large) ε is to begin with, we must be able to find N so that $n \geq N$ is enough to guarantee $|x_n - \ell|$ is smaller than ε.

It is not stated in the definition that ε has to be in any sense small, but it is not hard to see that any N that works for one ε will also work if you took a larger ε. So making ε smaller (while still keeping $\varepsilon > 0$) makes it more challenging to produce N that works.

You could think of it as a game you have to be sure you can win. Your opponent chooses $\varepsilon > 0$ and you have to be sure you can find an N that works no matter how malicious your opponent is in choosing really small $\varepsilon > 0$.

We have avoided deciding what small should be by allowing for any possible interpretation of 'small' distance. And we have avoided describing 'large' in any absolute way by saying only that it means 'all n larger than something'. We have to be able to come up with a suitable interpretation of large (a suitable $N \in \mathbb{N}$) no matter what $\varepsilon > 0$ is (no matter what 'small distance' is fixed for us).

Examples 2.6. *(i) The sequence* $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ *has limit* 0*.*

Proof. We are dealing with the sequence $(x_n)_{n=1}^{\infty}$ where $x_n = 1/n$ and $\ell = 0$. The aim then is to show

For each $\varepsilon > 0$ it is possible to find $N \in \mathbb{N}$ so that

$$\left|\frac{1}{n} - 0\right| < \varepsilon \text{ holds for every } n \in \mathbb{N} \text{ with } n \geq N.$$

In all these proofs we have to look carefully at what is required and then figure out how to do it.

Start with $\varepsilon > 0$ fixed (but we are not going to make any assumption about how it has been fixed — it is fixed for now but could be any positive quantity).

We want to ensure

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon$$

and we want to do this by making n large.

What we know (Proposition 1.18 (ii)) is that there is some $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$. Choose such an N. Then for any $n \geq N$ we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

Thus we have demonstrated that it is always possible to find $N \in \mathbb{N}$ as required. (Always, no matter what $\varepsilon > 0$ we start with.) This shows that the condition of the definition of limit is satisfied and so 0 is a limit of the sequence $(1/n)_{n=1}^{\infty}$. □

(ii) If $x_n = 1$ *for each* n *then 1 is a limit of the sequence* $(x_n)_{n=1}^{\infty} = (1)_{n=1}^{\infty}$ *(known as the constant sequence 1).*

Proof. Our aim is to show:

For each $\varepsilon > 0$ it is possible to find $N \in \mathbb{N}$ so that

$$|x_n - 1| < \varepsilon \text{ holds for every } n \in \mathbb{N} \text{ with } n \geq N.$$

As before, start with $\varepsilon > 0$ given but arbitrary. If we can find N for this ε we can do it for any $\varepsilon > 0$ because we have not assumed anything special about ε (other than $\varepsilon > 0$).

In this case $|x_n - 1| = |1 - 1| = 0 < \varepsilon$ for every n. We can just take $N = 1$. Then we have

$$|x_n - 1| = 0 < \varepsilon \text{ holds for every } n \geq 1$$

and the criterion in the definition of limit is satisfied. □

(iii) For $x_n = \frac{n^2}{n^2+1}$, *the sequence* $(x_n)_{n=1}^{\infty}$ *has limit 0.*

Proof. As before, start with $\varepsilon > 0$ given but arbitrary. Our aim is to show:

It is possible to find $N \in \mathbb{N}$ so that

$$|x_n - 1| < \varepsilon \text{ holds for every } n \in \mathbb{N} \text{ with } n \geq N.$$

Consider

$$|x_n - 1| = \left|\frac{n^2}{n^2+1} - 1\right| = \left|\frac{n^2 - (n^2+1)}{n^2+1}\right| = \left|\frac{-1}{n^2+1}\right| = \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n}$$

If we choose $N \in \mathbb{N}$ large enough so that $\frac{1}{N} < \varepsilon$ (as we know we can do by Proposition 1.18(ii)), we will then have for every $n \geq N$

$$|x_n - 1| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

and so

$$|x_n - 1| < \varepsilon \text{ for all } n \geq N.$$

We have shown how to produce $N \in \mathbb{N}$ that works (for the $\varepsilon > 0$ we began with).

Since $\varepsilon > 0$ is arbitrary we have established that we can find N no matter what $\varepsilon > 0$ is. So we have established that the criterion of the definition holds (for 1 to be a limit). □

Proposition 2.7. *A sequence* $(x_n)_{n=1}^\infty$ *of numbers can have at most one limit (in* $\mathbb{R}$*).*

Proof. The claim is then that if ℓ and ℓ' are two limits of the same sequence $(x_n)_{n=1}^\infty$, then $\ell = \ell'$.

The method of proof is to come up with a notion of 'close' that eliminates the possibility $\ell \neq \ell'$.

If $\ell \neq \ell'$ then $\varepsilon = |\ell - \ell'|/3$ is a valid choice (that is $\varepsilon > 0$). So if ℓ is a limit of the sequence, then we must be able to find $N \in \mathbb{N}$ so that

$$|x_n - \ell| < \varepsilon \text{ holds for all } n \geq N.$$

But, since ℓ' is also a limit, we must be able to find $N' \in \mathbb{N}$ so that

$$|x_n - \ell'| < \varepsilon \text{ holds for all } n \geq N'.$$

If we take $n = \max(N, N')$ then we have both

$$|x_n - \ell| < \varepsilon \text{ and } |x_n - \ell'| < \varepsilon.$$

Using the triangle inequality (on the distance between ℓ and ℓ' via x_n for this n) we get a contradiction as follows. We have

$$|\ell - \ell'| = |(\ell - x_n) + (x_n - \ell')| \leq |\ell - x_n| + |x_n - \ell'|$$

and so

$$|\ell - \ell'| \leq |x_n - \ell| + |x_n - \ell'| < \varepsilon + \varepsilon = \frac{2}{3}|\ell - \ell'|$$

But this is impossible as it implies (multiplying by $3/|\ell - \ell'| > 0$) that $3 < 2$.

This contradiction arose from $\ell \neq \ell'$. So we must have $\ell = \ell'$ and the sequence cannot have two different limits. □

Notation 2.8. Now that we know limits are unique (if they exist) we can be sure no confusion will arise if we introduce a notation

$$\lim_{n\to\infty} x_n$$

to stand for the limit ℓ of a convergent sequence. The point is that if there could be several limits, then using the same notation for them all could cause confusion or mislead you into thinking that the same number is meant when it is used a second time.

What is perhaps slightly dangerous is to use the notation $\lim_{n\to\infty} x_n$ when we are still unsure whether the sequence is convergent or not. So we may quite often use $\lim_{n\to\infty} x_n$ while discussing the existence of the limit or even write "$\lim_{n\to\infty} x_n$ does not exist" when we conclude that the sequence we are dealing with is not a convergent sequence.

The next theorem allows us to avoid using the definition of limit directly in many cases. We still need some examples which have been worked out via the definition, but we can use the theorem to work out many other limits.

Theorem 2.9. *Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be convergent sequences with $\lim_{n\to\infty} x_n = \ell_1$ and $\lim_{n\to\infty} x_n = \ell_2$. Then*

(i) $\lim_{n\to\infty}(x_n + y_n) = \ell_1 + \ell_2$

(In words: the limit of a sum is the sum of the limits (provided the individual limits exist).)

(ii) If $\alpha \in \mathbb{R}$ (constant), then $\lim_{n\to\infty} \alpha x_n = \alpha \ell_1$.

(Multiplying each term of a convergent sequence by a constant produces a new convergent sequence where the new limit is that constant times the original limit.)

(iii) $\lim_{n\to\infty}(x_n y_n) = \ell_1 \ell_2$

(The limit of a product is the product of the limits (provided the individual limits exist).)

(iv) If $\ell_2 \neq 0$ then there exists $n_0 \in \mathbb{N}$ so that $y_n \neq 0$ for all $n \geq n_0$ and also

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\ell_1}{\ell_2}$$

(The limit of a quotient is the quotient of the limits provided the denominator limit is not 0 (and the individual limits exist).)

Proof. (i) Start with $\varepsilon > 0$ given. Our aim is to find (or show that it is possible to find) $N \in \mathbb{N}$ so that

$$|(x_n + y_n) - (\ell_1 + \ell_2)| < \varepsilon \text{ holds for all } n \geq N.$$

We know that we can find $N_1 \in \mathbb{N}$ so that

$$|x_n - \ell_1| < \frac{\varepsilon}{2} \text{ holds for all } n \geq N_1$$

(using the definition of $\lim_{n\to\infty} x_n = \ell_1$). We can also find $N_2 \in \mathbb{N}$ so that

$$|y_n - \ell_2| < \frac{\varepsilon}{2} \text{ holds for all } n \geq N_2.$$

Take $N = \max(N_1, N_2)$ and then we can say that for any $n \geq N$ we must have both $N \geq N_1$ and $n \geq N_2$. So, for such n, we have both $|x_n - \ell_1| < \varepsilon/2$ and $|y_n - \ell_2| < \varepsilon/2$. it follows that, for $n \geq N$, we have

$$\begin{aligned} |(x_n + y_n) - (\ell_1 + \ell_2)| &= |(x_n - \ell_1) + (y_n - \ell_2)| \\ &\leq |x_n - \ell_1| + |y_n - \ell_2| \\ &\qquad \text{(using the triangle inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus we can find N as desired and we can do it no matter what $\varepsilon > 0$ we start with. So we have established (i).

(ii) First we dispose of the case $\alpha = 0$. In that case $\alpha x_n = 0$ is the constant sequence 0 and so its limit is $0 = \alpha \ell_1$. We have established (ii) for $\alpha = 0$.

When $\alpha \neq 0$, we start with $\varepsilon > 0$ given. Our aim is to show we can find $N \in \mathbb{N}$ so that

$$|\alpha x_n - \alpha \ell_1| < \varepsilon \text{ for all } n \geq N.$$

To do this we examine the quantity $|\alpha x_n - \alpha \ell_1|$ that is supposed to be 'small' (in the sense of $< \varepsilon$) for n large. Notice

$$|\alpha x_n - \alpha \ell_1| = |\alpha(x_n - \ell_1)| = |\alpha||x_n - \ell_1|.$$

We know we can find $N \in \mathbb{N}$ so that

$$|x_n - \ell_1| < \frac{\varepsilon}{|\alpha|} \text{ for all } n \geq N.$$

For this N we see that

$$|\alpha x_n - \alpha \ell_1| = |\alpha||x_n - \ell_1| < |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon \text{ for all } n \geq N.$$

Thus N has the desired property. As we can do this no matter what $\varepsilon > 0$ we start with, we have established (ii) for $\alpha \neq 0$.

(iii) We first show this for the case $\ell_1 = 0$. Using the fact that $\lim_{n\to\infty} y_n = \ell_2$, we claim first that there must be a number M so that $|y_n| \leq M$ for all n. Using the ε-N definition of the limit with 'ε' $= 1$ to find $N_0 \in \mathbb{N}$ so that

$$|y_n - \ell_2| < 1 \text{ for all } n \geq N_0.$$

It follows that, for $n \geq N_0$,

$$|y_n| = |(y_n - \ell_2) + \ell_2| \leq |y_n - \ell_2| + |\ell_2| < 1 + |\ell_2|$$

(using the triangle inequality). If we now take M to be the maximum of the finitely many numbers $|y_1|, |y_2|, \ldots, |y_{n_0-1}|, 1 + |\ell_2|$, we can see that $|y_n| \leq M$ for all n.

Using this fact, we can see how to guarantee that $|x_n y_n - \ell_1 \ell_2| = |x_n y_n - 0| = |x_n y_n|$ is small for n large.

Start with $\varepsilon > 0$ given. We know that $|x_n y_n - \ell_1 \ell_2| = |x_n||y_n| \leq M|x_n|$, and by the fact that $\lim_{n\to\infty} x_n = 0$ we can find $N \in \mathbb{N}$ so that

$$|x_n - \ell_1| = |x_n| < \frac{\varepsilon}{M} \text{ for all } n \geq N.$$

For this choice of N we have

$$|x_n y_n - \ell_1 \ell_2| = |x_n||y_n| \leq M|x_n| < M\frac{\varepsilon}{M} = \varepsilon \text{ for all } n \geq N.$$

Since we can find such N no matter what $\varepsilon > 0$ is given, we see that we have shown (iii) in the case $\ell_1 = 0$.

The general case (any value of ℓ_1 now) can be deduced from this because

$$x_n y_n = (x_n - \ell_1 + \ell_1) y_n = (x_n - \ell_1) y_n + \ell_1 y_n$$

By what we have just shown

$$\lim_{n\to\infty} (x_n - \ell_1) y_n = 0$$

(because $\lim_{n\to\infty}(x_n - \ell_1) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} \ell_1 = \ell_1 - \ell_1 = 0$ by part (i)). Also $\lim_{n\to\infty} \ell_1 y_n = \ell_1 \ell_2$ by (ii). So, using (i),

$$\lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} (x_n - \ell_1) y_n + \ell_1 y_n = 0 + \ell_1 \ell_2 = \ell_1 \ell_2.$$

(iv) First we establish that for n large $y_n \neq 0$ (using the assumption $\ell_2 \neq 0$).

Choose $\varepsilon = |\ell_2|/2$ in the ε-N definition of $\lim_{n\to\infty} y_n = \ell_2$. We can find $n_0 \in \mathbb{N}$ so that

$$|y_n - \ell_2| < \varepsilon = \frac{|\ell_2|}{2} \text{ for all } n \geq n_0.$$

But for such $n \geq n_0$ we then have

$$|y_n| = |(y_n - \ell_2) + \ell_2| \geq |\ell_2| - |y_n - \ell_2| > |\ell_2| - \frac{|\ell_2|}{2} = \frac{|\ell_2|}{2}$$

using a variation of the triangle inequality to be found in Exercises 2 (question 2(a)).

> From this we can see that x_n/y_n makes perfect sense for $n \geq n_0$. That leaves us with the question of what x_n/y_n would mean if $y_n = 0$. Of course, it would not mean anything for such n and it seems strange to refer to $\lim_{n\to\infty} x_n/y_n$ when there is a possibility that the quotient sequence $(x_n/y_n)_{n=1}^{\infty}$ is not properly specified. We could add the hypothesis that $y_n \neq 0$ for all n and strictly that is what we should do. However, the problem can only arise (if it arises at all) for $n < n_0$ and there are then only a finite number of such n. As far as the limit of a sequence is concerned, changing a finite number of terms will not affect the limit (because we can always ensure that the N we take in the definition of the limit is bigger than any finite number of n we want to avoid). On these grounds we could live with the doubt over how to deal with a finite number of terms x_n/y_n where $y_n = 0$.
>
> Usually that is done, but we could also just insist that y_n is always nonzero.

Now, we have not only showed that (from the assumptions made) $y_n \neq 0$ for $n \geq n_0$ (some n_0), we have shown more that there is a number $c = |\ell_2|/2 > 0$ and $n_0 \in \mathbb{N}$ so that $|y_n| > c$ for all $n \geq n_0$. This allows us to estimate (for $n \geq n_0$)

$$\left|\frac{1}{y_n} - \frac{1}{\ell_2}\right| = \left|\frac{\ell_2 - y_n}{y_n \ell_2}\right| = \frac{|\ell_2 - y_n|}{|y_n||\ell_2|} \leq \frac{|\ell_2 - y_n|}{c|\ell_2|}.$$

We can use this estimate to show $\lim_{n\to\infty} y_n = 1/\ell_2$. Start with $\varepsilon > 0$ given and choose $N_1 \in \mathbb{N}$ so that

$$|y_n - \ell_2| < c|\ell_2|\varepsilon \text{ for all } n > N_1$$

(as we know we can do because $c|\ell_2|\varepsilon > 0$ by the definition of $\lim_{n\to\infty} y_n = \ell_2$).

Now take $N = \max(n_0, N_1)$. For $n \geq N$ we have both $n \geq n_0$ and $n \geq N_1$, and therefore we have both

$$\left|\frac{1}{y_n} - \frac{1}{\ell_2}\right| \leq \frac{|\ell_2 - y_n|}{c|\ell_2|} \leq \frac{|y_n - \ell_2|}{c|\ell_2|} \text{ and } |y_n - \ell_2| < c|\ell_2|\varepsilon.$$

Thus, for all $n \geq N$ we have

$$\left|\frac{1}{y_n} - \frac{1}{\ell_2}\right| < \frac{c|\ell_2|\varepsilon}{c|\ell_2|} = \varepsilon.$$

Since we can find such $N \in \mathbb{N}$ for any given $\varepsilon > 0$, we have shown

$$\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{\ell_2}.$$

We can deduce (iv) from this using (iii) as follows

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} x_n \frac{1}{y_n} = \ell_1 \frac{1}{\ell_2} = \frac{\ell_1}{\ell_2}.$$

$\square$

Examples 2.10. *(i) For $x_n = \frac{n^2}{n^2+1}$, the sequence $(x_n)_{n=1}^\infty$ has limit 0.*

Proof. We did this same example already using a direct approach (the definition alone) and now we can redo it using the theorem above.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2}{n^2+1} &= \lim_{n\to\infty} \frac{n^2/n^2}{(n^2+1)/n^2} \\
&= \lim_{n\to\infty} \frac{1}{1+1/n^2} \\
&= \lim_{n\to\infty} \frac{1}{1+(1/n)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}(1+(1/n)^2)} \\
&\quad \text{(by Theorem 2.9(iv))} \\
&= \frac{1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty}(1/n)^2} \\
&\quad \text{(by Theorem 2.9(i))} \\
&= \frac{1}{1+(\lim_{n\to\infty} 1/n)^2} \\
&\quad \text{(by Theorem 2.9(iii))} \\
&= \frac{1}{1+(0)^2} = 1.
\end{aligned}
$$

□

(ii) Note that the same technique can be used to work out many similar examples such as

$$\lim_{n\to\infty} \frac{3n^4+2n^3-11n+12}{n^5+2n+1}$$

(start by dividing above and below by the largest power n^5).

(iii) For $x_n = (-1)^n$, the sequence $(x_n)_{n=1}^\infty$ has no limit (is not convergent).

Proof. The theorem is no help for this. We hae to go back to the definition of limit. Suppose on the contrary that $\lim_{n\to\infty}(-1)^n = \ell$ for some $\ell \in \mathbb{R}$. The idea is to choose an interpretation of 'close to ℓ' which could not be satisfied simultaneously by both 1 and -1. Take $\varepsilon = 1/2$ in the ε-N criterion and we deduce that there must be $N \in \mathbb{N}$ so that

$$|(-1)^n - \ell| < \varepsilon = 1/2 \text{ for all } n \geq N.$$

Choosing $n = N$ and $n = N+1$ (one of which must be odd and the other even) we deduce that both

$$|(-1) - \ell| < \frac{1}{2} \text{ and } |1 - \ell| < \frac{1}{2}.$$

Then from the triangle inequality we see we would have to have

$$|1-(-1)| = |(1-\ell) + (\ell-(-1))| \leq |1-\ell| + |\ell-(-1)| < \frac{1}{2} + \frac{1}{2} = 1$$

and this gives the contradiction $2 < 1$.

So there can be no limit ℓ for this sequence. □

Definition 2.11. *If* $(x_n)_{n=1}^\infty$ *is sequence, then it is called a* monotone increasing sequence *if* $x_n \leq x_{n+1}$ *holds for each* $n \in \mathbb{N}$.

This means that $x_1 \leq x_2 \leq x_3 \leq \cdots$*, and the term* non-decreasing *is perhaps more descriptive. If we mean really increasing, we use the term* strictly monotone increasing *— this means* $x_n < x_{n+1}$ *for each* $n \in \mathbb{N}$.

We can similarly define monotone decreasing sequence *(which means* $x_n \geq x_{n+1}$ *holds for each* $n \in \mathbb{N}$*. Again* non-increasing *may be a better term and the term* strictly monotone decreasing *is used for* $x_n > x_{n+1}$ *for* $n \in \mathbb{N}$ *(or* $x_1 < x_2 > x_3 > \cdots$.

A sequence $(x_n)_{n=1}^\infty$ *is called* monotone *if it is* either *monotone increasing* or *monotone decreasing.*

Examples 2.12. *The sequence* $(n^2)_{n=1}^\infty$ *is (strictly) monotone increasing while* $\left(\frac{1}{n}\right)_{n=1}^\infty$ *is (strictly) monotone decreasing.*

Any constant sequence $(c)_{n=1}^\infty$ *is both monotone increasing and monotone decreasing.*

The sequence $(x_n)_{n=1}^\infty$ *with* $x_n = (-1)^n$ *is* ***not*** *monotone.*

Definition 2.13. *If* $(x_n)_{n=1}^\infty$ *is sequence, then it is called* bounded above *if the set of values* $\{x_n : n \in \mathbb{N}\}$ *is bounded above. (That means, if there is* $u \in \mathbb{R}$ *so that* $x_n \leq u$ *holds for all* $n \in \mathbb{N}$*.)*

The sequence is called bounded below *if the set* $\{x_n : n \in \mathbb{N}\}$ *is bounded below.*

The sequence is called bounded *if it is* ***both*** *bounded above* ***and*** *bounded below.*

Theorem 2.14. *If a monotone sequence is bounded then it is convergent (has a limit in* $\mathbb{R}$*).*

Proof. We take the case of a monotone increasing bounded sequence (as the case if a monotone decreasing bounded sequence can be done in an analogous way or deduced from the fact that if $(x_n)_{n=1}^\infty$ is monotone decreasing and bounded then the sequence $(-x_n)_{n=1}^\infty$ is monotone increasing and bounded).

Let $(x_n)_{n=1}^\infty$ be monotone increasing and bounded (above). Let $u = \mathrm{lub}\{x_n : n \in \mathbb{N}\}$ (which exists by the least upper bound principle (P13) because the set is nonempty and bounded).

We show $\lim_{n\to\infty} x_n = u$.

Start with $\varepsilon > 0$ given. Then $u - \varepsilon < u =$ the least upper bound, and so cannot be an upper bound itself. To say it is **not** an upper bound means it is **not true** that

$x_n \leq u - \varepsilon$ holds for all $n \in \mathbb{N}$. But that means there is at least one $N \in \mathbb{N}$ where $x_N > u - \varepsilon$. As the sequence is monotone increasing $x_N \leq x_{N+1} \leq x_{N+2} \leq \cdots$, we can see that $x_n > u - \varepsilon$ for all $n \geq N$[1] Since $x_n \leq u =$ and upper bound, we have now

$$u - \varepsilon < x_n \leq u \text{ holds for all } n \geq N$$

It follows that $-varepsilon < x_n - u \leq 0$ and $|x_n - u| = -(x_n - u) < \varepsilon$ for all $n \geq N$.

As we can find such N for each $\varepsilon > 0$, we have verified $\lim_{n\to\infty} x_n = u$. □

By a (positive) decimal fraction we mean a number $0.d_1d_2d_3\ldots$ for a posssibly infinite sequence $d_1, d_2, \ldots$ of digits taken from $d_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. When the decimal terminates it means

$$0.d_1d_2\ldots d_n = \sum_{j=1}^{n} \frac{d_j}{10^j}$$

(and is actually a rational number). For infinite decimals we need a definition of what they mean. We define

$$0.d_1d_2\ldots = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{d_j}{10^j}$$

(if the limit exists — but it does always exist as we will show in a moment). We get arbitrary decimals by adding a whole number (positive integer) to a decimal of this type and then possibly multiply by -1 (to get neagtives).

Corollary 2.15. *Every decimal represents a real number.*

Proof. The point is that the limit $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{d_j}{10^j}$ always exists. But $x_n = \sum_{j=1}^{n} \frac{d_j}{10^j}$ is a monotone increasing sequence because

$$x_{n+1} = \sum_{j=1}^{n+1} \frac{d_j}{10^j} = x_n + \frac{d_{n+1}}{10^{n+1}} \geq x_n.$$

It is also bounded above because

$$x_n \leq \sum_{j=1}^{n} \frac{9}{10^j} = 0.99\ldots 9 \quad (n \text{ nines})$$

[1]We could show by induction on $j \in \mathbb{N}$ that $x_{N+j-1} > u - \varepsilon$ holds for all $j \in \mathbb{N}$.

and so $x_n \leq 1 - 0.00\ldots 1 < 1$.

So $\lim_{n\to\infty} x_n$ does exist by Theorem 2.14. $\square$

Example 2.16. *The decimal with repeating nines* $0.999\ldots = 1$.

Proof. Taking $d_j = 9$ for all j we see that

$$x_n = \sum_{j=1}^{n} \frac{d_j}{10^j} = \sum_{j=1}^{n} \frac{9}{10^j} = 0.99\ldots 9 = 1 - 0.00\ldots 1 = 1 - \frac{1}{10^n}$$

Since $10^n > n$ we can show quite easily that $\lim_{n\to\infty} \frac{1}{10^n} = 0$ and $\lim_{n\to\infty} x_n = 1$. $\square$

Definition 2.17. *If* $(x_n)_{n=1}^{\infty}$ *is sequence, then by a* subsequence *we mean a sequence* $(y_j)_{j=1}^{\infty}$ *which is of the form* $y_j = x_{n_j}$ *where* $n_1 < n_2 < n_3 < \cdots$ *is a strictly monotone increasing sequence of natural numbers).*

To be really precise the subsequence is not just the sequence $y_1, y_2, y_3, \ldots$ but also the map $j \mapsto n_j$ which determines the indices n_j used to pick out the subsequence.

This if $x_n = (-1)^n$, then the constant sequence with $y_j = 1$ for $j = 1, 2, \ldots$ is a subsequence in lost of different ways. For example $n_j = 2j$ works but so does $n_j = 4j$. The fact that the values same $1, 1, 1, \ldots$ can be picked out in more than one way is almost never an issue.

For $x_n = 1/n$, the sequence with terms $y_j = 1/(j^2)$ is a subsequence (with $n_j = j^2$).

Proposition 2.18. *If* $\lim_{n\to\infty} x_n = \ell$ *and* $(x_{n_j})_{j=1}^{\infty}$ *is any subsequence, then* $\lim_{j\to\infty} x_{n_j} = \ell$

Proof. An observation we will use is that if $(n_j)_{j=1}^{\infty}$ is a strictly monotone increasing sequence of natural numbers, then $n_j \geq j$ for each $j \in \mathbb{N}$. This we can check by induction. $n_1 \geq 1$ is true. Assuming $n_j \geq j$ i true, we can deduce from $n_j < n_{j+1}$ and the fact that $n_j, n_{j+1} \in \mathbb{N}$ that $n_j + 1 \leq n_{j+1}$. Hence $j + 1 \leq n_j + 1 \leq n_{j+1}$.

Now let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} x_n = \ell$ we can find $N \in \mathbb{N}$ so that

$$|x_n - \ell| < \varepsilon \text{ holds for all } n \geq N.$$

But then $j \geq N$ implies $n_j \geq j \geq N$ and so $|x_{n_j} - \ell| < \varepsilon$. This shows $\lim_{j\to\infty} x_{n_j} = \ell$. $\square$

Theorem 2.19. *(Bolzano Weierstrass Theorem) If $(x_n)_{n=1}^\infty$ is a bounded sequence in $\mathbb{R}$, then it has a convergent subsequence (a subsequence with a limit in $\mathbb{R}$).*

The proof is mostly based on the following lemma.

Lemma 2.20. *Every sequence $(x_n)_{n=1}^\infty$ has a monotone subsequence.*

Proof. The key idea in the proof is the idea of a *peak point* for the sequence (the term peaking index might be better). We say that $n \in \mathbb{N}$ is a peak point of the sequence $(x_n)_{n=1}^\infty$ if it is true that

$$x_n \geq x_m \text{ for all } m \in \mathbb{N} \text{ with } m \geq n.$$

Either the sequence $(x_n)_{n=1}^\infty$ has infinitely many peak points or it has only a finite number.

If it has infinitely many, let $n_1 =$ the smallest peak point, $n_2 =$ the smallest peak point with $n_2 > n_1$ and so on. We define n_{j+1} inductively (or recursively) as the smallest peak point where $n_{j+1} > n_j$. Then $x_{n_j} \geq x_{n_{j+1}}$ because n_j is a peak point. Hence $(x_{n_j})_{j=1}^\infty$ is a monotone decreasing subsequence of $(x_n)_{n=1}^\infty$.

If on the other hand there are only a finite number of peak points, there is $n_1 \in \mathbb{N}$ with $n_1 >$ all peak points. Since n_1 is not itself a peak point, there is $n_2 > n_1$ with $x_{n_1} < x_{n_2}$. Next n_2 is not a peak point (bigger than all peak points) and so there is $n_3 > n_2$ with $x_{n_2} < x_{n_3}$. In this way we can pick $n_1 < n_2 < n_3 < \cdots$ with $n_j < n_{j+1}$ and $x_{n_j} < x_{n_{j+1}}$ for each $j \in \mathbb{N}$. $(x_{n_j})_{j=1}^\infty$ is a monotone increasing subsequence. $\square$

Proof. (of Theorem 2.19) Starting with a bounded sequence $(x_n)_{n=1}^\infty$, we use Lemma 2.20 to find a monotone subsequence $(x_{n_j})_{j=1}^\infty$.

Because the original sequence is bounded, there are numbers $L, U \in \mathbb{R}$ so that $L \leq x_n \leq U$ for each $n \in \mathbb{N}$. It follows that $L \leq x_{n_j} \leq U$ for each $j \in \mathbb{N}$ and so the subsequence $(x_{n_j})_{j=1}^\infty$ is bounded as well as monotone. By Theorem 2.14 $\lim_{j\to\infty} x_{n_j}$ exists in $\mathbb{R}$. $\square$