Chapter 1: Numbers 121 2004–05

As explained in the introduction, rather than trying to build up (real) numbers from (say) the natural numbers, or at the other extreme just assuming we know what they are, we will write down a list of properties or axioms that we will assume the real numbers satisfy.

Many of the properties are so simple they may seem almost too simple to mention. However, we will get a fairly concise list of properties that could be verified if one were to construct the real numbers out of natural numbers, integers and rationals. It is in fact also possible to prove that there is essentially only one set of objects satisfying the full list of properties for real numbers. We will neither construct the numbers nor prove this statement that our list of axioms characterises the real numbers.

We will assume that there is a set of objects (which we call the set of *real numbers*) denoted by the symbol $\mathbb{R}$ and we will list properties we assume about $\mathbb{R}$.

Properties of addition

We assume that the set $\mathbb{R}$ comes with an operation called addition, which associates to each pair of $a \in \mathbb{R}$ and $b \in \mathbb{R}$ another number $a + b \in \mathbb{R}$. Further we assume that the following properties hold for this operation

(P1) (associative law)

$$(a + b) + c = a + (b + c)$$

whenever $a, b, c \in \mathbb{R}$.

(P2) (existence of an additive identity)

There is a number $0 \in \mathbb{R}$ with the properties

$$\begin{aligned} a + 0 &= a \\ 0 + a &= a \end{aligned}$$

for each $a \in \mathbb{R}$.

(P3) (existence of additive inverses)

For each $a \in \mathbb{R}$ there is a $b \in \mathbb{R}$ with the properties

$$\begin{aligned} a + b &= 0 \\ b + a &= 0 \end{aligned}$$

(P4) (commutativity of addition, also known as the abelian property)

$$a + b = b + a \text{ for each } a, b \in \mathbb{R}$$

Remark 1.1. *You will notice that this is quite a short list of properties about addition, and certainly all will seem obviously true. They are all true in* $\mathbb{Z}$ *and* $\mathbb{Q}$ *as well as in* $\mathbb{R}$*. Later you will see other situations where there is a notion of* $+$ *and we have these properties also (vectors can be added, as can matrices, for example).*

If we stuck with $\mathbb{N}$ *we would not have (P2) or (P3). For* $\mathbb{N}_0$ *we would not have (P3) (additive inverses).*

Lemma 1.2. *There is only one zero element in* $\mathbb{R}$*.*

That is, if $\tilde{0}$ *and* $\tilde{\tilde{0}}$ *are two elements of* $\mathbb{R}$ *satisfying (P2), then* $\tilde{0} = \tilde{\tilde{0}}$*.*

Proof. Writing out in detail what we are assuming, we are assuming

$$a + \tilde{0} = a \text{ and } \tilde{0} + a = a \text{ for each } a \in \mathbb{R}$$

and also

$$a + \tilde{\tilde{0}} = a \text{ and } \tilde{\tilde{0}} + a = a \text{ for each } a \in \mathbb{R}.$$

If we look at

$$\tilde{0} + \tilde{\tilde{0}}$$

using the property of $\tilde{0}$ we get $\tilde{0} + \tilde{\tilde{0}} = \tilde{\tilde{0}}$, but if we look at it using the property of $\tilde{\tilde{0}}$ we get $\tilde{0} + \tilde{\tilde{0}} = \tilde{0}$. So

$$\tilde{\tilde{0}} = \tilde{0} + \tilde{\tilde{0}} = \tilde{0}.$$

□

Note 1.3. *One consequence of (P2) is that the set of numbers is not empty. There is at last a number* 0*.*

Lemma 1.4. *If $a \in \mathbb{R}$ then there is only one additive inverse for a.*

That is if $b \in \mathbb{R}$ satisfies

$$a + b = 0 \textit{ and } b + a = 0$$

and if $c \in \mathbb{R}$ satisfies

$$a + c = 0 \textit{ and } c + a = 0,$$

then $b = c$.

Proof. Consider

$$(c + a) + b = c + (a + b)$$

(true by associativity (P1)). Working it out we get

$$0 + b = c + 0$$

and so

$$b = c$$

by the property (P2) of zero. □

Note 1.5. *Because of the lemma, we are justified in having a special notation $-a$ for the additive inverse of $a \in \mathbb{R}$.*

Without the lemma, we could fall into a trap if $-a$ was something that could be differently interpreted in different places.

Properties of multiplication

As well as addition, we also assume there is another operation called multiplication on $\mathbb{R}$ and associating to each $a \in \mathbb{R}$ and $b \in \mathbb{R}$ another number $a \cdot b \in \mathbb{R}$. [For a while we will write in the $\cdot$ where we mean multiplication, but later we will use the standard way of writing ab instead of $a \cdot b$.]

We assume that multiplication satisfies the following properties:

(P5) (associative law)

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

whenever $a, b, c \in \mathbb{R}$.

(P6) (existence of a multiplicative identity)

There is a number $1 \in \mathbb{R}$ with $1 \neq 0$ and the properties

$$\begin{aligned} a \cdot 1 &= a \\ 1 \cdot a &= a \end{aligned}$$

for each $a \in \mathbb{R}$.

(P7) (existence of multiplicative inverses)

For each $a \in \mathbb{R}$ with $a \neq 0$ there is a $b \in \mathbb{R}$ with the properties

$$\begin{aligned} a \cdot b &= 1 \\ b \cdot a &= 1 \end{aligned}$$

(P8) (commutativity of multiplication)

$$a \cdot b = b \cdot a \text{ for each } a, b \in \mathbb{R}$$

(P9) (distributive law, connecting $+$ and $\cdot$)

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

and

$$(a + b) \cdot c = a \cdot c + b \cdot c$$

holds for each $a, b, c \in \mathbb{R}$.

Lemma 1.6. *(i)* $0 \cdot x = 0$ *for each* $x \in \mathbb{R}$

(ii) $-x = (-1) \cdot x$ *for each* $x \in \mathbb{R}$

Proof. (i) Fix $x \in \mathbb{R}$ and let $y = 0 \cdot x$. Then

$$\begin{aligned} y + y &= 0 \cdot x + 0 + \cdot x \\ &= (0 + 0) \cdot x \qquad \text{(distributive law (P9))} \\ &= 0 \cdot x \\ &= y \end{aligned}$$

Adding $-y$ to both sides of $y + y = y$ we get

$$\begin{aligned} (y+y)+(-y) &= y+(-y) \\ y+(y+(-y)) &= 0 \quad \text{(associativity of $+$ and definition of $-y$)} \\ y+0 &= 0 \\ y &= 0 \end{aligned}$$

So we have $y = 0 \cdot x = 0$ as required.

(ii) Fix $x \in \mathbb{R}$ and consider $x + (-1) \cdot x = 1 \cdot x + (-1) \cdot x = (1 + (-1)) \cdot x$ (distributive law (P9)). But $(1 + (-1)) \cdot x = 0 \cdot x = 0$ by the first part. So we get $x + (-1) \cdot x = 0$. This means that $(-1) \cdot x \in \mathbb{R}$ has the property that additive inverses have and so $(-1) \cdot x$ is $-x$.

□

Notation 1.7. *From now we allow ourselves to write $x-y$ when we mean $x+(-y)$.*

Lemma 1.8. *(i) The following cancellation law holds in $\mathbb{R}$:*

$$x, y, z \in \mathbb{R}, x + z = y + z \Rightarrow x = y$$

(ii) The following cancellation law holds in $\mathbb{R}$:

$$x, y, z \in \mathbb{R}, z \neq 0, x \cdot z = y \cdot z \Rightarrow x = y$$

Proof. Exercise. □

Example 1.9. *Here is a strange example of a set with operations $+$ and $\cdot$ where all the properties (P1) to (P9) hold for elements of the set. The idea behind this example is to show that we do not have enough properties to say we are really dealing with familiar numbers.*

Let $S = \{e, o\}$ (and think of e as standing for *even* and o for *odd*). Define the operations by the following two tables.

$+$	e	o
e	e	o
o	o	e

$\cdot$	e	o
e	e	e
o	e	o

You can check that each of the properties (P1) to (P9) hold when $\mathbb{R}$ is replaced by S everywhere. (This means checking each of the finitely many possibilities for $a, b, c \in S$. The rôle of 0 is taken by $e \in S$ and the rôle of 1 by $o \in S$.)

The properties (P1) to (P9) are called the *axioms for a field* or the *field axioms*. Abstractly, any sent with operations $+$ and $\cdot$ given in such a way that the properties (P1) to (P9) hold is called a *field*.

The example just given is a field with two elements (the smallest possible number because there have to be $0 \neq 1$ in any field). You may also come across essentially the same example when you study the integers modulo 2. What you do there is to replace every integer by its remainder after dividing by 2, so that we end up with $\{0, 1\}$ and the operations $+$ and $\cdot$ are as follows: perform the operation in $\mathbb{Z}$ and ten take the remainder on dividing by 2.

Remark 1.10. We proceed now to introduce properties about ordering of numbers (which are not satisfied by the example $S = \{e, o\}$). We are familiar with the idea of "(strictly) less than".

We recall that $x < y$ is the notation for "less than" and that $x \leq y$ (reads "less that or equal") means that *either* $x < y$ *or* $x = y$. $x > y$ is just another way of writing $y < x$ and we also have $x \geq y$ ("greater than or equal").

Rathe than introducing a relation $<$ (with some associated rules) we introduce a concept of (strictly) positive. If we understand what it means to be strictly positive and we will later get $x < y$ by defining it to mean $y - x$ positive.

Properties of (strictly) positive numbers

We suppose that there is given a set $P \subset \mathbb{R}$ (called the set of *positive* numbers) with the following properties

(P10) (Trichotomy law) For each number $x \in \mathbb{R}$ one and only one of the following statements is true:

(i) $x = 0$

(ii) $x \in P$

(iii) $-x \in P$

(P11) (sums of 'positive' numbers are 'positive')

If $x, y \in P$, then $x + y \in P$.

(P12) (products of 'positive' numbers are 'positive')

If $x, y \in P$, then $x \cdot y \in P$.

Notation 1.11. We say $x \in \mathbb{R}$ is (strictly) *positive* if $x \in P$, while if $-x \in P$ we say that x is (strictly) *negative*.

The trichotomy law then says that a give $x \in \mathbb{R}$ is either 0, positive or negative (and cannot satisfy two of these 3 options).

Lemma 1.12. *(i) If $-a \in P$ and $-b \in P$, then $ab \in P$ (the product of any two negative numbers is positive).*

(ii) If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 \in P$.

(iii) $1 \in P$.

Proof. (i) By Lemma 1.6 (ii), if $-a \in P$ and $-b \in P$, then

$$(-a)(-b) = ((-1)a)(-b) = (a(-1))(-b) = a((-1)(-a))$$

by commutativity and associativity, and so

$$(-a)(-b) = a(-(-b)) = ab$$

(ii) If $a \in \mathbb{R}$ and $a \neq 0$, then by trichotomy we either have $a \in P$ or $-a \in P$. If $a \in P$ then $a^2 \in P$ by (P12). If $-a \in P$, then $a^2 = aa \in P$ by part (i).

(iii) From part (ii), since $1 \neq 0$, we have $1^2 \in P$. But $1^2 = 1$ and so we have $1 \in P$.

$\square$

Definition 1.13 (Less than). *We define a relation $<$ on $\mathbb{R}$ (called (strictly)* less than*) by $x < y$ (for $x, y \in \mathbb{R}$) if $y - x \in P$.*

Proposition 1.14. *(a) (trichotomy in another form) If $x, y \in \mathbb{R}$, then one and only one of the following statements is true*

(i) $x = y$

(ii) $x < y$

(iii) $y < x$

(b) (transitivity) If $x, y, z \in \mathbb{R}$, $x < y$ and $y < z$, then $x < z$.

(c) If $x, y, z \in \mathbb{R}$ and $x < y$, then $x + z < y + z$.

(d) If $x, y, z \in \mathbb{R}$, $x < y$ and $0 < z$, then $xz < yz$.

Proof. (a) follows by applying (P10) to $y - x$.

(b) If $x < y$ and $y < z$, then $y - x \in P$ and $z - y \in P$. Hence their sum $(y - x) + (z - y) = z - x \in P$ by (P11). So $x < z$.

(c) If $x < y$, then we have $y - x \in P$. For any $z \in \mathbb{R}$, $x + z < y + z$ means $(y + z) - (x + z) = y - x \in P$ and we are assuming this is true.

(d) If $x < y$ and $0 < z$, then we have $y - x \in P$ and $z \in P$. So, by (P12), $(y - x)z \in P$. Thus $yz - xz \in P$ or $xz < yz$.

□

Note 1.15. We write 2 for $1+1$. Since $1 \in P$ we have $0 < 1$ and so $0+1 < 1+1$ or $1 < 2$. Thus $2 \neq 1$ in $\mathbb{R}$.

We can continue in this vein and write 3 for $2 + 1$, 4 for $3 + 1$ and this way find a copy of the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ inside $\mathbb{R}$.

From this we can find $\mathbb{Z}$ and $\mathbb{Q}$ inside $\mathbb{R}$.

However we cannot necessarily get more than $\mathbb{Q}$ because the rational numbers $\mathbb{Q}$ with the usual interpretation of addition, multiplication and positive (elements p/q with $p, q \in \mathbb{N}$) does satisfy all of (P1) – (P12).

We want an additional property (or properties) that ensures that, for example there is a number $\sqrt{2}$ in $\mathbb{R}$. We will do this with one additional property, but it takes a bit more digesting that the properties (P1) – (P12) (which all seem more or less obvious when you see them stated).

We can picture the real numbers as points on an axis (where we pick out an origin marked with the number 0 and a point 1 — then we mark 2 twice as far from 0 as 1 in the same direction, $\frac{1}{2}$ as the midpoint between 0 and 1, and so on). We can geometrically mark off a distance $\sqrt{2}$ from the origin in the direction 1 and then we want a property that ensures there is a number at that point on the line. If we had only the points corresponding to rational numbers $\mathbb{Q}$, there would be no number corresponding to the point in question. Our last property is going to say that all the points correspond to numbers, but the way of formulating this will be somewhat indirect.

A direct way to go would be to consider decimal expansions. Recall that $\sqrt{2} = 1.4142\ldots$. The decimal approximations 1, $1.4 = 1 + \frac{4}{10} = \frac{14}{10}$, $1.41 =$

$1 + \frac{4}{10} + \frac{1}{10^2} = \frac{141}{100}$, *etc.*, should be heading towards some number that is really there and not 'missing' from $\mathbb{R}$.

It would perhaps be possible to insist that every (finite or infinite) decimal represents a real number, but this can be messy because for example $0.999\ldots$ (repeating 9's) is the same as 1.

Our approach is to approach the matter in a way that seems almost backwards. Instead of demanding that we get somewhere by building up 1, 1.4, 1.41, 1.414, 1.4142, ... from below, we go at it from the other side and work down towards the right number.

Definition 1.16. *If $S \subset \mathbb{R}$ is a subset, then a number $u \in \mathbb{R}$ is called an upper bound for S if it is true that $s \leq u$ for each $s \in S$.*

If we picture $\mathbb{R}$ as a number line and S as a marked set of points on the line, then to say that u is an upper bound for S means that every element of S is to the 'left' of u, or at least none of them are definitely to the right of S.

Example 1.17. (a) Let $S = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \text{ and } x \leq 1\}$ = the closed interval from 0 to 1 (inclusive).

Then $u = 2$ is an upper bound for S because $s \in S \Rightarrow s \leq 1 \Rightarrow s \leq 2$.

Also $u = 1$ is an upper bound. As well as being an upper bound $1 \in S$ and so here we have 1 as a greatest element of S.

(b) $S = (-\infty, -2) = \{x \in \mathbb{R} : x < -2\}$ = the open interval to the 'left' of -2 has $u = -1$ as an upper bound.

It also has $u = -2$ as an upper bound, the "most efficient" upper bound in some sense. But there is no greatest element in S.

(c) The empty set $S = \emptyset$ is a subset of $\mathbb{R}$. Any number $u \in \mathbb{R}$ is an upper bound for S in this case.

The reason for that is that it **is true** that

$$s \leq u \text{ holds for each } s \in S.$$

The reason it is true is that there are no elements $s \in S$ and so there are no elements we need to check the inequality about. We say that the statement is *vacuously* true.

We now state our last property for $\mathbb{R}$.

Least upper bound principle

(P13) If a set $S \subset \mathbb{R}$ in *nonempty* and has an upper bound, then it has a least upper bound.

By a *least upper bound* for a set $S \subset \mathbb{R}$ we mean an upper bound $u \in \mathbb{R}$ for S with the property that $u \leq u'$ for each upper bound u' of S.

Proposition 1.18. *(i) $\mathbb{N} \subset \mathbb{R}$ has no upper bound in $\mathbb{R}$*

(ii) If $x \in \mathbb{R}$ and $0 < x$, then there is $n \in \mathbb{N}$ with $\frac{1}{n} < x$.

Proof. (i) If $\mathbb{N} \subset \mathbb{R}$ had an upper bound in $\mathbb{R}$, then (because $1 \in \mathbb{N}$ and so $\mathbb{N} \neq \emptyset$) the least upper bound principle says that there would have to be an upper bound $u \in \mathbb{R}$ for $\mathbb{N}$.

Now $u - 1$ cannot be an upper bound for $\mathbb{N}$ because $u - 1 < u$ and u is supposed to be $\leq$ every upper bound. To say that $u - 1$ is *not* an upper bound means that it is *not rue to say* that $n \leq u - 1$ holds for each $n \in \mathbb{N}$. To say that is not true means that there *is at least one* $n \in \mathbb{N}$ where $n > u - 1$.

Then $n + 1 > u$. But $n + 1 \in \mathbb{N}$ as well as n and this contradicts the assertion that u is an upper bound for $\mathbb{N}$.

The contradiction came about because we assumed there was an upper bound for $\mathbb{N}$ and the conclusion we draw is that $\mathbb{N}$ has no upper bound in $\mathbb{R}$.

(ii) If $x \in \mathbb{R}$, $x > 0$, then $\frac{1}{x} \in \mathbb{R}$ and so cannot be an upper bound for $\mathbb{N}$ (by the first part). So there is some $n \in \mathbb{N}$ with $n > \frac{1}{x}$. It follows (for example by multiplying both sides by $\frac{x}{n}$) that $x > \frac{1}{n}$.

□

The first part of the proposition can be thought of as saying that there are no 'infinitely large' real numbers.

Proposition 1.19. *There is a number $x \in \mathbb{R}$ with $x^2 = 2$.*

Proof. Consider $S = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < 2\}$. Notice that $S \neq \emptyset$ because $1 \in S$. It is bounded above by 2. To see this suppose on the contrary $x \in S$ and $x > 2$. Then $x^2 > 2x > 2^2 = 4 > 2$, contradicting $x^2 < 2$. We conclude that $x \leq 2$ for each $x \in S$, that is that 2 is an upper bound for S.

Now the least upper bound principle states that there is a least upper bound $u \in \mathbb{R}$ for S. Our claim is that $u^2 = 2$. If $u^2 \neq 2$ there are two other possibilities (i) $u^2 < 2$ and (ii) $u^2 > 2$. We now show that neither (i) nor (ii) are the case.

If $u^2 < 2$, we claim that there is $n \in \mathbb{N}$ with $u + \frac{1}{n} \in S$ (and this would contradict u being an upper bound). Now

$$\left(u + \frac{1}{n}\right)^2 = u^2 + 2\frac{u}{n} + \frac{1}{n^2} \leq u^2 + 2\frac{u}{n} + \frac{1}{n} = u^2 + \frac{2u+1}{n}.$$

It follows that $\left(u + \frac{1}{n}\right)^2 < 2$ if $u^2 + \frac{2u+1}{n} < 2$ or, equivalently if $\frac{2u+1}{n} < 2 - u^2$. Since $u \geq 1 \in S$, we have $u > 0$ and so $2u + 1 > 0$. Thus $\frac{2u+1}{n} < 2 - u^2$ is equivalent to $\frac{1}{n} < \frac{2-u^2}{2u+1}$. We can find an $n \in \mathbb{N}$ with this property by the proposition above. For such $n \in \mathbb{N}$ we then have $\left(u + \frac{1}{n}\right)^2 < 2$ and $u + \frac{1}{n} > u > 0$. Hence $u + \frac{1}{n} \in S$ contradicting u an upper bound for S. So (i) is eliminated.

If $u^2 > 2$ we claim that there is $n \in \mathbb{N}$ so that $u - \frac{1}{n}$ is an upper bound for S (smaller than the least upper bound S and so a contradiction). We choose $n \in \mathbb{N}$ so that $\left(u - \frac{1}{n}\right)^2 > 2$. This we can do because

$$\left(u - \frac{1}{n}\right)^2 = u^2 - 2\frac{u}{n} + \frac{1}{n^2} > u^2 - 2\frac{u}{n}$$

and this will be > 2 we ensure

$$u^2 - 2\frac{u}{n} > 2$$

or, equivalently

$$u^2 - 2 > 2\frac{u}{n}.$$

We can arrange this by choosing $n \in \mathbb{N}$ (via the proposition above) so that or $\frac{1}{n} < \frac{u^2-2}{2u}$. This is possible because $u > 0$ and we are assuming $u^2 - 2 > 0$.

Having chosen n so that $\left(u - \frac{1}{n}\right)^2 > 2$ we claim now that $u - \frac{1}{n}$ is an upper bound for S. If not there $x \in S$ with $x > u - \frac{1}{n}$. Since $u \geq 1 \in S$ and $1/n \leq 1$ we have $u - \frac{1}{n} \geq 0$. So

$$x^2 > x\left(u - \frac{1}{n}\right) > \left(u - \frac{1}{n}\right)^2 > 2$$

contradicting $x \in S$.

So now we have shown that there is an upper bound $u - 1/n$ for S strictly smaller than the supposed least upper bound u. Thus (ii) is not possible.

Having eliminated the possibilities (i) and (ii), we are left with $u^2 = 2$ as the only remaining possibility. $\square$

Remark 1.20. A useful remark is that a set $S \subset \mathbb{R}$ can only have one least upper bound. If u is a least upper bound, then we have $u \leq u'$ for each upper bound u'. If u_1 is also a least upper bound then also $u_1 \leq u'$ for each upper bound u'. So taking $u' = u_1$ in the property of u we have

$$u \leq u_1$$

and taking $u' = u$ in the property of u_1 we have

$$u_1 \leq u.$$

It follows that $u = u_1$.

So we can refer to *the* least upper bound of a set $S \subset \mathbb{R}$ if $S \neq \emptyset$ and S has some upper bound.

There is a **greatest lower bound principle** as well as a least upper bound principle, but one follows from the other.

First a *lower bound* for a set $S \subset \mathbb{R}$ is a number $\ell \in \mathbb{R}$ satisfying $\ell \leq s$ for each $s \in S$. A greatest lower bound for a set $S \subset \mathbb{R}$ is a lower bound ℓ for S with the property that $\ell' \leq \ell$ for each upper bound ℓ' for S. As with upper bounds, there can only be one least upper bound for a set S (or none).

There is a way to flip between upper and lower bounds (but not for the same set). If ℓ is an upper bound for $S \subset \mathbb{R}$ (that is $\ell \leq s$ for each $s \in S$) then $-\ell$ is an upper bound for the set $T = \{-s : s \in S\}$ (that is $-s \leq -\ell$ for each $s \in S$, or $t \leq -\ell$ for each $t \in T$). One can also check that ℓ is a greatest lower bound for S exactly when $-\ell$ is a least upper bound for T, and this is why the least upper bound principle tells us that the following is true:

> **Greatest lower bound principle:** If $S \subset \mathbb{R}$ is nonempty and has a lower bound, then it has a greatest lower bound.

We say that a subset $S \subset \mathbb{R}$ is *bounded above* if there is some upper bound for S. Also S is called *bounded below* if there is some lower bound for S.

A set $S \subset \mathbb{R}$ is *bounded* if it is *both* bounded above and bounded below.

When a subset $S \subset \mathbb{R}$ has a least upper bound we will often write $\mathrm{lub}(S)$ for that least upper bound. Another notation in common use is $\sup(S)$, where $\sup$ is an abbreviation for the word *supremum*.

When a subset $S \subset \mathbb{R}$ has a greatest lower bound we will often write glb(S) for that greatest lower bound. Another notation in common use is $\inf(S)$, where $\inf$ is an abbreviation for the word *infimum*.

To make life complicated, we may sometimes refer to lub(S), $\sup(S)$ or $\inf(S)$ when we are discussing the possible existence of these for a particular S. We might write 'find $\sup(S)$ or show that it does not exist' instead of 'find the least upper bound for S or show that there is no least upper bound'.

Definition 1.21. *The* absolute value $|x|$ *of* $x \in \mathbb{R}$ *is defined by the rule*

$$|x| = \begin{cases} x & \textit{if } x \geq 0 \\ -x & \textit{if } x < 0 \end{cases}$$

Example 1.22. $|-3| = -(-3) = 3$

Note 1.23. $|x| \geq 0$ *for each* $x \in \mathbb{R}$*.*

Theorem 1.24 (Triangle inequality). *If* $x, y \in \mathbb{R}$*, then* $|x + y| \leq |x| + |y|$*.*

Proof. The proof relies on considering the 4 separate cases

1. $x \geq 0$ and $y \geq 0$
2. $x < 0$ and $y < 0$
3. $x \geq 0$ and $y < 0$
4. $x < 0$ and $y \geq 0$

(and these cover all possibilities for x and y).

Here are the arguments

1. If $x \geq 0$ and $y \geq 0$, then $x + y \geq 0$ and so
$$|x + y| = x + y = |x| + |y|.$$
That is equality holds and so $|x + y| \leq |x| + |y|$ is valid.
2. If $x < 0$ and $y < 0$, then $x + y < 0$ and so
$$|x + y| = -(x + y) = (-x) + (-y) = |x| + |y|$$
and again equality holds.

3. If $x \geq 0$ and $y < 0$, we consider two sub cases

   (a) $x + y \geq 0$
   Then we have
   $$|x + y| = x + y$$
   and $|x| + |y| = x + (-y)$. The desired inequality $|x + y| \leq |x| + |y|$ then reduces to $x + y \leq x - y$, which is equivalent (adding $-x$ to both sides) to $y \leq -y$ and this is in turn equivalent to $2y \leq 0$ — which we know to be true in this case.

   (b) $x + y < 0$
   We have then $|x + y| = -(x + y) = -x - y$ and again $|x| + |y| = x + (-y)$. The desired inequality $|x + y| \leq |x| + |y|$ then reduces to $-x - y \leq x - y$ and this is equivalent to $-x \leq x$ or $0 \leq 2x$ — which we know to be true in this case.

4. The case $x < 0$ and $y \geq 0$ is similar to the previous case (if we swop the rôles of x and y).

$\square$

Remark 1.25. We can help to see our way by looking at suitable pictorial or graphical representations of what we are doing. (But we do not want to rely on the pictures for the proof as it is sometimes possible to draw pictures that are misleading or are over simplified.)

We can think of points $x \in \mathbb{R}$ as points on a number line, and $|x|$ as the 'distance' from x to the origin on the line. For $a, b \in \mathbb{R}$ we can think of $|a - b|$ as the distance between a and b.

If $a, b, c \in \mathbb{R}$ and we take $x = a - b$, $y = b - c$ in the triangle inequality 1.24 then we get
$$|(a - b) + (b - c)| \leq |a - b| + |b - c|.$$
This simplifies to
$$|a - c| \leq |a - b| + |b - c|, \tag{1}$$
or
$$\text{distance}(a, c) \leq \text{distance}(a, b) + \text{distance}(b, c)$$
The 'triangles' we can see on a line are degenerate (or flattened) triangles, but the latter way of looking at it says that the length of one side of a triangle is at most the sum of the lengths of the other two sides.

We can recover Theorem 1.24 from knowing (1) holds for all $a, b, c \in \mathbb{R}$. Fix $x, y \in \mathbb{R}$ any two numbers and take $a = x$, $b = 0$ and $c = -y$ in (1). This means that Theorem 1.24 is equivalent to the statement that (1) holds for all $a, b, c \in \mathbb{R}$ (and explains the name of the thorem).

Remark 1.26. *As mentioned previously, we will not prove that there is a system of numbers where all the properties (P1) to (P13) are satisfied. It is possible to prove it, but it is laborious and maybe not all that enlightening.*

It is further possible to prove that there can really only be one such system of numbers. Given two such systems, satisfying all of (P1) to (P13), there has to be a way to match up the numbers in one system with the numbers in the other so that all the things we consider (addition, multiplication, positive) match between one system and the other. Again, we will not try to prove that this is true.

We will just take $\mathbb{R}$ (with its properties) as given and work with it.