Chapter 0: Introduction 121 2004–05

Before we start in earnest, I'd like to give some kind of preview and introduction to the course.

'Analysis' means, more or less, calculus from a theoretical point of view. We will not be concerned very much with how to do new things, but more about revisiting topics you have heard about before. During our new visit, we will aim to explain why everything works as it does in quite some detail. Our explanations will be logical and develop more or less everything from scratch. You can use your experience of calculus as a guide to where we are going, but we now want to prove that everything works.

We will need to start somewhere, and then develop everything from there. We could start with almost nothing (say, just the language of sets, subsets, elements and so on) and develop everything from there. However, this takes rather too long and we will instead start with (real) numbers as more or less understood. However, we will be fairly careful about even that, and our approach is to start with a list of key properties we expect numbers to have. Many of them will be so simple you might wonder why we bother to state them, but some are a little less obvious. In the end we will have a list of properties of the real numbers that cannot be satisfied by any other system of numbers.

If we were to start with sets only, we would develop the *natural numbers*

$$\mathbb{N} = \{1, 2, 3, \ldots\},$$

the number *zero*, the *natural numbers including 0*

$$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\},$$

the *integers* (or whole numbers)

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$$

and the *rational numbers*

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

My notation for $\mathbb{N}$ and $\mathbb{N}_0$ is maybe not always used (and some books would have 0 in $\mathbb{N}$, while using some other notation for the positive integers — you might need to check what their notation is when you pick up a book).

Now, our first point is that the rational numbers are not enough. The first 'proof' we give is that $\sqrt{2}$ is not a rational number.

To state it this way is a bit odd, because we might interpret it as follows. Say rational numbers are all we know about. Then there is not going to be a number $\sqrt{2}$ and so we are on the one hand referring to a number $\sqrt{2}$ and then showing there is no such number. To avoid such a paradox, we can state the result like this:

Proposition 0.1. *There is no number* $\frac{p}{q} \in \mathbb{Q}$ *with*

$$\left(\frac{p}{q}\right)^2 = 2$$

There are some other ways to phrase it, maybe equally as good or a little better that the above. Even if you think they are no better, it can help to have differnt ways to sty things. It can help you to think about what the statement means, or to understand the point.

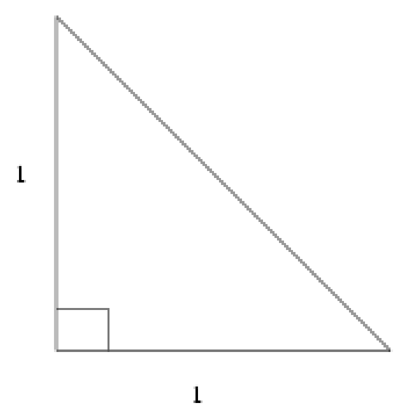
A rational number p/q implies (at least in our notation) $p, q \in \mathbb{Z}$ and $q \neq 0$. We did *not* make that really clear above and so perhaps we should have said:

> There is no $r \in \mathbb{Q}$ with $r^2 = 2$.

or, we could have phrased it like this:

> There are no integers $p, q \in \mathbb{Z}$ with $q \neq 0$ and $p^2 = 2q^2$.

There is a geomerical way to look at the statement. It is sometimes a help to be able to visualise statements. Suppose we construct a right triangle where both legs are one unit in length (or the same length).



According to Pythagorus' theorem the (ratio of the) lenghth of the hypoteneuse (to the lenghths of the sides) must be $\sqrt{2}$. Our proposition is saying that we cannot construct this same length by taking p times the length of the sides and diving that in q equal parts.

This fact was discovered over 2000 years ago by the Greek geometers and it disturbed them greatly.

Proof. (or the proposition). We will prove the statement in the formulation that there are no integers $p, q \in \mathbb{Z}$ with $q \neq 0$ and $p^2 = 2q^2$. It will help to keep in mind that $p^2 = 2q^2$ is the same as $(p/q)^2 = 2$ as long as we rule out $q = 0$.

We will start be assuming that we managed to find $p, q \in \mathbb{Z}$ with $q \neq 0$ and $p^2 = 2q^2$. Our aim then is to show that this could not be so.

> **Aside:** The punchline of the proof is based on ideas relating to even and odd numbers. If $p^2 = 2q^2$, then p^2 must be even. But, as squares of odd numebrs are still odd, this means that p has to be even, or divisible by 2. Therefore p^2 is actually divisible by $2^2 = 4$. As $p^2 = 2q^2$ this means that q^2 has to be even. So q has to be even as well as q. Now, there is no real problem with having p and q both even, but it does mean that the fraction $\frac{p}{q}$ could be 'simplified' by dividing above and below by 2.
>
> What we will do then to make this into a punchline is to arrange p and q to that all common factors (of 2 at least) have been divided out of p/q.

The integer p we start with might or might not be divisible by 2. If it is divisible by 2 we can write p as 2 times a whole number and that whole number might be divisible by 2 again (or not). However, we cannot be able to divide p by 2 indefintely unless $p = 0$. Since we are assuming $p = 2q^2$ and $q \neq 0$, we cannot have $p = 0$. So, after dividing out as many powers of 2 as possible from p we end up with

$$p = 2^n p_1$$

where $n \geq 0$ and p_1 are integers and p_1 is odd. Similarly we can write

$$q = 2^m q_1 \qquad (m, q_1 \in \mathbb{Z}, m \geq 0, q_1 \text{ odd}).$$

Thus

$$\frac{p}{q} = \frac{2^n p_1}{2^m q_1}$$

$$
\begin{aligned}
&= \begin{cases} \frac{2^{n-m}p_1}{q_1} & \text{if } n \geq m \\ \frac{p_1}{2^{m-n}q_1} & \text{if } n < m \end{cases} \\
&= \frac{p_2}{q_2}
\end{aligned}
$$

where at least one of p_2, q_2 is odd. From $p^2 = 2q^2$ it follows that $p_2^2 = 2q_2^2$.

Now we can say that p_2^2 is even, hence p_2 must be even (squares of odd numbers are still odd). Thus p_2^2 is divisible by $2^2 = 4$ and so $2q_2^2$ is divisible by 4. So q_2^2 is divisible by 2, and so q_2 must be even. Now we have concluded that p_2 and q_2 are both even, contradicting what we arranged — that at least one of p_2 and q_2 is odd.

The reason we got to this contradiction was that we assumed we could find $p, q \in \mathbb{Z}$ so that $p^2 = 2q^2$ and $q \neq 0$. The only way to avoid this contradiction is that there are no such integers p, q (or no sqaure root of 2 in $\mathbb{Q}$). $\square$

Exercise 0.2. *Show that there is no square root of 3 in* $\mathbb{Q}$.