Computation versus formulae for norms of elementary operators

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ElOp2009

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Hence can define $T|_I \colon I \to I$ and $T^I \colon A/I \to A/I$.

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Coroll. algorithm to check that $\|\delta_c\| = 2\|c - \lambda\|$ (if λ 'known')

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$(x, z) \mapsto (x + |\lambda|^2 - 2\Re(z\bar{\lambda}), z - \lambda)$ maps $W(c^*c, c)$ to $W((c - \lambda)^*(c - \lambda), c - \lambda)$

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Fact: infimum is attained (without increasing ℓ).

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Now some algorithms. (from [IJ2003])

Recall $u = \sum_{i=1}^{\ell} a_i \otimes b_i \in \mathcal{B}(H) \otimes \mathcal{B}(H)$, $T = \theta(u)$, $Tx = \sum_{j=1}^{\ell} a_j x b_j$. $\mathbf{a} = [a_1, a_2, \ldots, a_\ell]$, $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^t$.

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What about $\|T\|$?

Trick: reduce computation of $\|T\|$ to computation of cb norms of 'rank one' elem. ops where $\|\cdot\|_{cb} = \|\cdot\|$.

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$$\|\delta_c\| = \sup \mathsf{tgm}\left(\begin{bmatrix} \langle cc^*\xi, \xi\rangle & \langle c\xi, \xi\rangle \\ \langle c^*\xi, \xi\rangle & 1 \end{bmatrix}, \begin{bmatrix} 1 & -\langle c\eta, \eta\rangle \\ -\langle c^*\eta, \eta\rangle & \langle c^*c\eta, \eta\rangle \end{bmatrix}\right)$$

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For $\phi_1, \phi_2 \in P(A)$ say $\phi_1 \sim \phi_2$ if $[\pi_{\phi_1}] = [\pi_{\phi_2}]$ in $\hat{A}$.

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$$\|T\| = \sup\{\mathbf{tgm}(Q(\mathbf{a}^*, \phi_1), Q(\mathbf{b}, \phi_2)) : \phi_1, \phi_2 \in P(A), \phi_1 \sim \phi_2\}$$

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Example: Equality in $\|T\|_{cb} \leq \left\| \sum_{j=1}^{\ell} a_j \otimes b_j \right\|_h$?

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Corollary: $\left\|\sum_{j=1}^{\ell} a_j \otimes b_j\right\|_h = \sup\{\mathsf{tgm}(Q(\mathbf{a}^*, \psi_1), Q(\mathbf{b}, \psi_2)) : \psi_1, \psi_2 \in \mathsf{co}(P(A))\}$

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Lemma. [AST2009-TAMS] $Q \subset A$ primal, $\psi_1, \psi_2 \in S(A/Q)$ implies $\exists$ commonly indexed nets $(\phi_{1,\alpha})_\alpha$ and $(\phi_{2,\alpha})_\alpha$ in $F_f(A)$ so that $\phi_{1,\alpha} \to \psi_1$ and $\phi_{2,\alpha} \to \psi_2$.

Proposition.

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Thm. θ_Z isometric $\iff$ $\text{Min-Primal}(A) = \mathrm{Glimm}(A)$.

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Similar results for Θ_Z: $M(A) \otimes_{Z,h} M(A) \to CB(A)$ with conditions on Glimm ideals of $M(A)$.

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Similar results for $\Theta_Z \colon M(A) \otimes_{Z,h} M(A) \to CB(A)$ with conditions on Glimm ideals of $M(A)$.

Proposition $T \in \mathcal{E}\ell(A)$,
$Tx = \sum_{j=1}^{\ell} a_j x b_j \Rightarrow \|T\|_{cb} \leq \sqrt{\ell}\|T\|$.