PYU44T20 Quantum Optics and Information
Problem Set 3 due 25/04/2023

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SS Theoretical Physics

Contents

1 Analytical Exercises

(a)

$$\hat{A} \to \hat{A} \otimes \mathbb{1} \qquad\qquad \hat{B} \to \mathbb{1} \otimes \hat{B}$$

$$\begin{aligned}
\left[\hat{A} \otimes \mathbb{1}, \mathbb{1} \otimes \hat{B}\right] &= \left(\hat{A} \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes \hat{B}\right) - \left(\mathbb{1} \otimes \hat{B}\right)\left(\hat{A} \otimes \mathbb{1}\right) \\
&= \hat{A}\mathbb{1} \otimes \mathbb{1}\hat{B} - \mathbb{1}\hat{A} \otimes \hat{B}\mathbb{1} \\
&= \hat{A} \otimes \hat{B} - \hat{A} \otimes \hat{B} \\
&= 0 \implies \hat{A} \otimes \mathbb{1} \text{ and } \mathbb{1} \otimes \hat{B} \text{ always commute}
\end{aligned}$$

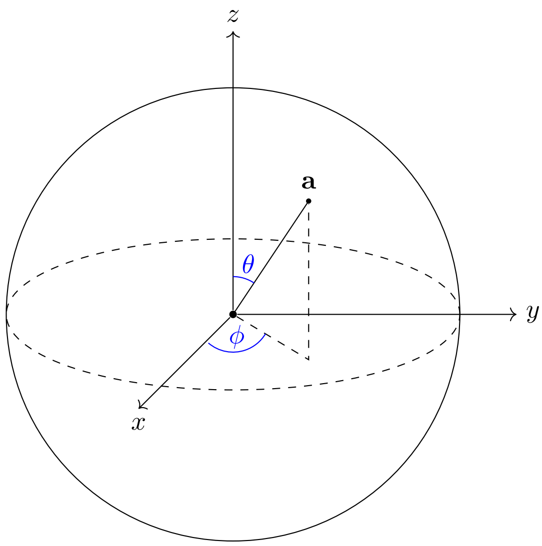
(b)

$$\hat{s}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{s}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{s}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned}
\hat{s}_x \left|\uparrow\right\rangle &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{\hbar}{2}\left|\downarrow\right\rangle \\
\hat{s}_y \left|\uparrow\right\rangle &= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 \\ i \end{pmatrix} \\
&= \frac{i\hbar}{2}\left|\downarrow\right\rangle \\
\hat{s}_z \left|\uparrow\right\rangle &= \frac{\hbar}{2}\begin{pmatrix} 10 & \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\left|\uparrow\right\rangle
\end{aligned}
\qquad
\begin{aligned}
\hat{s}_x \left|\downarrow\right\rangle &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{\hbar}{2}\left|\uparrow\right\rangle \\
\hat{s}_y \left|\downarrow\right\rangle &= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} -i \\ 0 \end{pmatrix} \\
&= -\frac{i\hbar}{2}\left|\uparrow\right\rangle \\
\hat{s}_z \left|\downarrow\right\rangle &= \frac{\hbar}{2}\begin{pmatrix} 10 & \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix} \\
&= -\frac{\hbar}{2}\left|\downarrow\right\rangle
\end{aligned}$$

$$\begin{aligned}
[\hat{s}_x, \hat{s}_y]\left|\uparrow\right\rangle &= \hat{s}_x\hat{s}_y\left|\uparrow\right\rangle - \hat{s}_y\hat{s}_z\left|\uparrow\right\rangle \\
&= \hat{s}_x\frac{i\hbar}{2}\left|\downarrow\right\rangle - \hat{s}_y\frac{\hbar}{2}\left|\downarrow\right\rangle \\
&= \frac{i\hbar^2}{2}\left|\uparrow\right\rangle + \frac{i\hbar^2}{2}\left|\uparrow\right\rangle \\
&= i\hbar^2\left|\uparrow\right\rangle \\
i\hbar\hat{s}_z\left|\uparrow\right\rangle &= i\hbar^2\left|\uparrow\right\rangle \\
\implies [\hat{s}_x, \hat{s}_y]\left|\uparrow\right\rangle &= i\hbar\hat{s}_z\left|\uparrow\right\rangle
\end{aligned}
\qquad
\begin{aligned}
[\hat{s}_x, \hat{s}_y]\left|\downarrow\right\rangle &= \hat{s}_x\hat{s}_y\left|\downarrow\right\rangle - \hat{s}_y\hat{s}_z\left|\downarrow\right\rangle \\
&= \hat{s}_x\left(-\frac{i\hbar}{2}\left|\uparrow\right\rangle\right) - \hat{s}_y\frac{\hbar}{2}\left|\uparrow\right\rangle \\
&= -\frac{i\hbar^2}{2}\left|\downarrow\right\rangle - \frac{i\hbar^2}{2}\left|\downarrow\right\rangle \\
&= -i\hbar^2\left|\downarrow\right\rangle \\
i\hbar\hat{s}_z\left|\downarrow\right\rangle &= -i\hbar^2\left|\downarrow\right\rangle \\
\implies [\hat{s}_x, \hat{s}_y]\left|\downarrow\right\rangle &= i\hbar\hat{s}_z\left|\downarrow\right\rangle
\end{aligned}$$

$$\begin{aligned}
\implies [\hat{s}_x, \hat{s}_y]\left(\alpha\left|\uparrow\right\rangle + \beta\left|\downarrow\right\rangle\right) &= i\hbar\hat{s}_z\left(\alpha\left|\uparrow\right\rangle + \beta\left|\downarrow\right\rangle\right) \\
\therefore [\hat{s}_x, \hat{s}_y] &= i\hbar\hat{s}_z
\end{aligned}$$



The states on the Bloch sphere represent normalised states of a qubit, similarly to how points on a unit sphere are a unit distance from the origin. These states in the qubit Hilbert space $\mathcal{H}_q$ can be parameterised as a point on the unit sphere S^2 in terms of spherical or Cartesian coordinates. For example,

$$\mathcal{H}_q \ni |{\uparrow}\rangle \sim (1, 0, \phi)_{\text{sph}} = (0, 0, 1)_{\text{Cart}} \in S^2$$
$$\mathcal{H}_q \ni |{\downarrow}\rangle \sim (1, \pi, \phi)_{\text{sph}} = (0, 0, -1)_{\text{Cart}} \in S^2$$

$$\mathcal{H}_q \ni |+\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle + |{\downarrow}\rangle\right) \sim \left(1, \frac{\pi}{2}, 0\right)_{\text{sph}} = (1, 0, 0)_{\text{Cart}} \in S^2$$
$$\mathcal{H}_q \ni |-\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle - |{\downarrow}\rangle\right) \sim \left(1, \frac{\pi}{2}, \pi\right)_{\text{sph}} = (-1, 0, 0)_{\text{Cart}} \in S^2$$

$$\mathcal{H}_q \ni |{\rightarrow}\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle + i\,|{\downarrow}\rangle\right) \sim \left(1, \frac{\pi}{2}, \frac{\pi}{2}\right)_{\text{sph}} = (0, 1, 0)_{\text{Cart}} \in S^2$$
$$\mathcal{H}_q \ni |{\leftarrow}\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle - i\,|{\downarrow}\rangle\right) \sim \left(1, \frac{\pi}{2}, \frac{3\pi}{2}\right)_{\text{sph}} = (0, -1, 0)_{\text{Cart}} \in S^2$$

It is important to note that the states on the Bloch sphere are simply visual representations of normalised states of a qubit, where the states at the poles of the x-, y-, and z-axes represent eigenstates of the respective Pauli matrices, and do not follow the same rules as vectors on a sphere. For example,

$$\langle{\uparrow}\,|\,{\downarrow}\rangle = 0 \neq (0, 0, 1)_{\text{Cart}} \cdot (0, 0, -1)_{\text{Cart}} = -1$$
$$\langle{\uparrow}\,|\,+\rangle = \frac{1}{\sqrt{2}} \neq (0, 0, 1)_{\text{Cart}} \cdot (1, 0, 0)_{\text{Cart}} = 0$$

(c)

$$
\begin{aligned}
|\uparrow\uparrow\rangle &= |\uparrow\rangle \otimes |\uparrow\rangle \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
|\uparrow\downarrow\rangle &= |\uparrow\rangle \otimes |\downarrow\rangle \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
\qquad
\begin{aligned}
|\downarrow\uparrow\rangle &= |\downarrow\rangle \otimes |\uparrow\rangle \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
|\downarrow\downarrow\rangle &= |\downarrow\rangle \otimes |\downarrow\rangle \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

$$\hat{S}_z = \hat{s}_z \otimes \mathbb{1} + \mathbb{1} \otimes \hat{s}_z$$

$$
\begin{aligned}
\hat{S}_z |\uparrow\uparrow\rangle &= \hat{s}_z |\uparrow\rangle \otimes \mathbb{1} |\uparrow\rangle + \mathbb{1} |\uparrow\rangle \otimes \hat{s}_z |\uparrow\rangle \\
&= \frac{\hbar}{2} |\uparrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes \frac{\hbar}{2} |\uparrow\rangle \\
&= \hbar |\uparrow\uparrow\rangle \\
\hat{S}_z |\uparrow\downarrow\rangle &= \hat{s}_z |\uparrow\rangle \otimes \mathbb{1} |\downarrow\rangle + \mathbb{1} |\uparrow\rangle \otimes \hat{s}_z |\downarrow\rangle \\
&= \frac{\hbar}{2} |\uparrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes -\frac{\hbar}{2} |\downarrow\rangle \\
&= 0
\end{aligned}
\qquad
\begin{aligned}
\hat{S}_z |\downarrow\uparrow\rangle &= \hat{s}_z |\downarrow\rangle \otimes \mathbb{1} |\uparrow\rangle + \mathbb{1} |\downarrow\rangle \otimes \hat{s}_z |\uparrow\rangle \\
&= -\frac{\hbar}{2} |\downarrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes \frac{\hbar}{2} |\uparrow\rangle \\
&= 0 \\
\hat{S}_z |\downarrow\downarrow\rangle &= \hat{s}_z |\downarrow\rangle \otimes \mathbb{1} |\downarrow\rangle + \mathbb{1} |\downarrow\rangle \otimes \hat{s}_z |\downarrow\rangle \\
&= -\frac{\hbar}{2} |\downarrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes -\frac{\hbar}{2} |\downarrow\rangle \\
&= -\hbar |\downarrow\downarrow\rangle
\end{aligned}
$$

Thus $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ are eigenstates of $\hat{S}_z$ with corresponding eigenvalues $\{\hbar, 0, 0, -\hbar\}$.

(d)

$$
\begin{aligned}
\hat{H} &= J \hat{s}_1 \cdot \hat{s}_2 \\
&= J \left(\hat{s}_{1,x} \hat{s}_{2,x} + \hat{s}_{1,y} \hat{s}_{2,y} + \hat{s}_{1,z} \hat{s}_{2,z} \right) \\
&= J \left[(\hat{s}_x \otimes \mathbb{1})(\mathbb{1} \otimes \hat{s}_x) + (\hat{s}_y \otimes \mathbb{1})(\mathbb{1} \otimes \hat{s}_y) + (\hat{s}_z \otimes \mathbb{1})(\mathbb{1} \otimes \hat{s}_z) \right] \\
&= J \left(\hat{s}_x \otimes \hat{s}_x + \hat{s}_y \otimes \hat{s}_y + \hat{s}_z \otimes \hat{s}_z \right)
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{H}, \hat{S}_z \right] &= \left[J \left(\hat{s}_x \otimes \hat{s}_x + \hat{s}_y \otimes \hat{s}_y + \hat{s}_z \otimes \hat{s}_z \right), \hat{s}_z \otimes \mathbb{1} + \mathbb{1} \otimes \hat{s}_z \right] \\
\frac{1}{J} \left[\hat{H}, \hat{S}_z \right] &= \left[\hat{s}_x \otimes \hat{s}_x, \hat{s}_z \otimes \mathbb{1} \right] + \left[\hat{s}_x \otimes \hat{s}_x, \mathbb{1} \otimes \hat{s}_z \right] \\
&\quad + \left[\hat{s}_y \otimes \hat{s}_y, \hat{s}_z \otimes \mathbb{1} \right] + \left[\hat{s}_y \otimes \hat{s}_y, \mathbb{1} \otimes \hat{s}_z \right] \\
&\quad + \left[\hat{s}_z \otimes \hat{s}_z, \hat{s}_z \otimes \mathbb{1} \right] + \left[\hat{s}_z \otimes \hat{s}_z, \mathbb{1} \otimes \hat{s}_z \right] \\
&= \hat{s}_x \hat{s}_z \otimes \hat{s}_x - \hat{s}_z \hat{s}_x \otimes \hat{s}_x + \hat{s}_x \otimes \hat{s}_x \hat{s}_z - \hat{s}_x \otimes \hat{s}_z \otimes \hat{s}_x \\
&\quad + \hat{s}_y \hat{s}_z \otimes \hat{s}_y - \hat{s}_z \hat{s}_y \otimes \hat{s}_y + \hat{s}_y \otimes \hat{s}_y \hat{s}_z - \hat{s}_y \otimes \hat{s}_z \otimes \hat{s}_y \\
&\quad + \hat{s}_z \hat{s}_z \otimes \hat{s}_z - \hat{s}_z \hat{s}_z \otimes \hat{s}_z + \hat{s}_z \otimes \hat{s}_z \hat{s}_z - \hat{s}_z \otimes \hat{s}_z \otimes \hat{s}_z \\
&= -\left[\hat{s}_z, \hat{s}_x \right] \otimes \hat{s}_x - \hat{s}_x \otimes \left[\hat{s}_z, \hat{s}_x \right] + \left[\hat{s}_y, \hat{s}_z \right] \otimes \hat{s}_y + \hat{s}_y \otimes \left[\hat{s}_y, \hat{s}_z \right] + 0 \\
&= -i\hbar \hat{s}_y \otimes \hat{s}_x - \hat{s}_x \otimes i\hbar \hat{s}_y + i\hbar \hat{s}_x \otimes \hat{s}_y + \hat{s}_y \otimes i\hbar \hat{s}_x \\
&= 0
\end{aligned}
$$

$$\implies \left[\hat{H}, \hat{S}_z \right] = 0$$

As $\hat{H}$ and $\hat{S}_z$ commute, we expect that they have some common eigenstates. $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are non-degenerate eigenstates of $\hat{S}_z$, and so should also be eigenstates of $\hat{H}$. $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are degenerate eigenstates of $\hat{S}_z$ with eigenvalue 0, and so some linear combinations of these should also be eigenstates of $\hat{H}$. Therefore, the eigenstates of $\hat{H}$ can be given by $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \alpha_1 |\uparrow\downarrow\rangle + \beta_1 |\downarrow\uparrow\rangle, \alpha_2 |\uparrow\downarrow\rangle + \beta_2 |\downarrow\uparrow\rangle\}$, where α_i, β_i are to be determined.

(e)

$$\hat{\sigma}_x \otimes \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\sigma}_y \otimes \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\sigma}_z \otimes \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
\hat{H} &= J\left(\hat{s}_x \otimes \hat{s}_x + \hat{s}_y \otimes \hat{s}_y + \hat{s}_z \otimes \hat{s}_z\right) \\
&= J\left(\frac{\hbar}{2}\hat{\sigma}_x \otimes \frac{\hbar}{2}\hat{\sigma}_x + \frac{\hbar}{2}\hat{\sigma}_y \otimes \frac{\hbar}{2}\hat{\sigma}_y + \frac{\hbar}{2}\hat{\sigma}_z \otimes \frac{\hbar}{2}\hat{\sigma}_z\right) \\
&= \frac{\hbar^2 J}{4}\left[\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] \\
&= \frac{\hbar^2 J}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

We can find the eigenvalues for the eigenstates $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ by acting $\hat{H}$ on each state.

$$\begin{aligned}
\hat{H}|\uparrow\uparrow\rangle &= \frac{\hbar^2 J}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{\hbar^2 J}{4}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{\hbar^2 J}{4}|\uparrow\uparrow\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{H}|\downarrow\downarrow\rangle &= \frac{\hbar^2 J}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{\hbar^2 J}{4}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{\hbar^2 J}{4}|\downarrow\downarrow\rangle
\end{aligned}$$

The eigenstates $\{\alpha_1 |\uparrow\downarrow\rangle + \beta_1 |\downarrow\uparrow\rangle, \alpha_2 |\uparrow\downarrow\rangle + \beta_2 |\downarrow\uparrow\rangle\}$ can be determined by acting $\hat{H}$ on $\alpha|\uparrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle$ and solving for α, β.

$$\begin{aligned}
\hat{H}\left(\alpha|\uparrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle\right) &= \frac{\hbar^2 J}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \alpha \\ \beta \\ 0 \end{pmatrix} \\
&= \frac{\hbar^2 J}{4}\begin{pmatrix} 0 \\ -\alpha + 2\beta \\ 2\alpha - \beta \\ 0 \end{pmatrix}
\end{aligned}$$

By inspection, $\alpha = \beta = \frac{1}{\sqrt{2}}$ and $\alpha = -\beta = \frac{1}{\sqrt{2}}$ correspond to normalised eigenstates:

$$\hat{H}\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) = \frac{\hbar^2 J}{4}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \frac{\hbar^2 J}{4}\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$\hat{H}\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) = \frac{\hbar^2 J}{4}\begin{pmatrix} 0 \\ -\frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ 0 \end{pmatrix} = -\frac{3\hbar^2 J}{4}\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

Thus $\left\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)\right\}$ are eigenstates of $\hat{H}$ with corresponding eigenvalues $\left\{\frac{\hbar^2 J}{4}, \frac{\hbar^2 J}{4}, \frac{\hbar^2 J}{4}, -\frac{3\hbar^2 J}{4}\right\}$.

(f)

$$\langle \Delta X_1^2\rangle + \langle \Delta X_2^2\rangle = \langle X_1^2\rangle - \langle X_1\rangle^2 + \langle X_2^2\rangle - \langle X_2\rangle^2$$

$$\begin{aligned}\langle X_1\rangle &= \langle n|\frac{1}{2}\left(\hat{a}+\hat{a}^\dagger\right)|n\rangle \\ &= \frac{1}{2}\left(\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^\dagger|n\rangle\right) \\ &= \frac{1}{2}\left(\sqrt{n+1}\langle n+1\,|\,n\rangle + \sqrt{n+1}\langle n\,|\,n+1\rangle\right) \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle X_2\rangle &= \langle n|\frac{1}{2i}\left(\hat{a}-\hat{a}^\dagger\right)|n\rangle \\ &= \frac{1}{2i}\left(\langle n|\hat{a}|n\rangle - \langle n|\hat{a}^\dagger|n\rangle\right) \\ &= \frac{1}{2i}\left(\sqrt{n+1}\langle n+1\,|\,n\rangle - \sqrt{n+1}\langle n\,|\,n+1\rangle\right) \\ &= 0\end{aligned}$$

$$\implies \langle \Delta X_1^2\rangle + \langle \Delta X_2^2\rangle = \langle X_1^2\rangle + \langle X_2^2\rangle$$

$$\begin{aligned}\langle X_1^2\rangle + \langle X_2^2\rangle &= \langle n|\frac{1}{2}\left(\hat{a}+\hat{a}^\dagger\right)\frac{1}{2}\left(\hat{a}+\hat{a}^\dagger\right)|n\rangle + \langle n|\frac{1}{2i}\left(\hat{a}-\hat{a}^\dagger\right)\frac{1}{2i}\left(\hat{a}-\hat{a}^\dagger\right)|n\rangle \\ &= \frac{1}{4}\langle n|\left(\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}\right)|n\rangle + \frac{1}{-4}\langle n|\left(\hat{a}^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}\right)|n\rangle \\ &= \frac{1}{4}\langle n|\left(\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}-\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}-\hat{a}^{\dagger 2}\right)|n\rangle \\ &= \frac{1}{2}\langle n|\left(\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\right)|n\rangle \\ &= \frac{1}{2}\left[\left(\hat{a}^\dagger|n\rangle\right)^\dagger\hat{a}^\dagger|n\rangle + \left(\hat{a}|n\rangle\right)^\dagger\hat{a}|n\rangle\right] \\ &= \frac{1}{2}\left[\left(\sqrt{n+1}|n+1\rangle\right)^\dagger\sqrt{n+1}|n+1\rangle + \left(\sqrt{n}|n-1\rangle\right)^\dagger\sqrt{n}|n-1\rangle\right] \\ &= \frac{1}{2}\left[(n+1)\langle n+1\,|\,n+1\rangle + n\langle n-1\,|\,n-1\rangle\right] \\ &= \frac{n+1+n}{2} \\ &= n+\frac{1}{2}\end{aligned}$$

$$\implies \langle \Delta X_1^2\rangle + \langle \Delta X_2^2\rangle = n+\frac{1}{2}$$

The position and momentum operators for a Harmonic oscillator are given by

$$\hat{X} = \eta \left(\hat{a}^\dagger + \hat{a} \right), \qquad \hat{P} = \frac{i\hbar}{2\eta} \left(\hat{a}^\dagger - \hat{a} \right), \qquad \eta \equiv \sqrt{\frac{\hbar}{2m\omega}},$$

and we have

$$\left\langle \hat{X}^2 \right\rangle = 2\eta^2 \left(n + \frac{1}{2} \right), \quad \left\langle \hat{P}^2 \right\rangle = \frac{\hbar^2}{2\eta^2} \left(n + \frac{1}{2} \right), \quad \left\langle \hat{X}^2 + \left(\frac{\eta}{\hbar} \hat{P} \right)^2 \right\rangle = \left(2\eta^2 + \frac{1}{2} \right) \left(n + \frac{1}{2} \right).$$

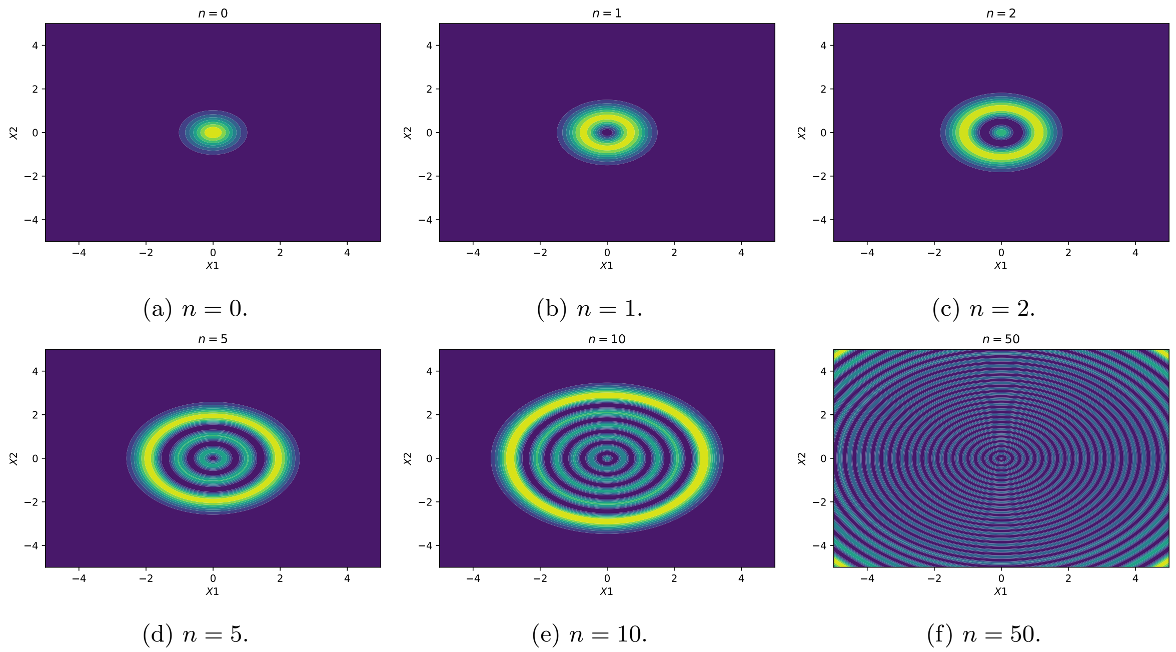
The wavefunction for the n-th excited state is given by

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} H_n \left(\frac{x}{\eta\sqrt{2}} \right) \frac{1}{\sqrt{\eta\sqrt{2\pi}}} \exp\left(-\frac{x^2}{4\eta^2} \right),$$

where $H_n(x)$ is the n-th Hermite polynomial. We can see that, for $\eta = \frac{1}{2}$, $\hat{X}_1 = \hat{X}$ and $\hat{X}_2 = \frac{\eta}{\hbar}\hat{P}$. Therefore the probability distribution in $X_1 X_2$-space is given by

$$\begin{aligned} p_n(X_1, X_2) &= \left| \psi_n \left(\sqrt{X_1^2 + X_2^2} \right) \right|^2 \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{2^n n!} e^{-2\left(X_1^2 + X_2^2\right)} \left[H_n \left(\sqrt{2\left(X_1^2 + X_2^2\right)} \right) \right]^2 \end{aligned}$$

Figure 1 shows distributions for the vacuum state ($n = 0$) and some excited states ($n > 0$).



(a) $n = 0$. (b) $n = 1$. (c) $n = 2$.

(d) $n = 5$. (e) $n = 10$. (f) $n = 50$.

Figure 1: Plots of the probability distribution of the quadrature operators in $X1X2$-space.

(g)

$$\begin{aligned}
\langle\alpha|\,\hat{E}(z,t)\,|\alpha\rangle &= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(\alpha^*)^n}{\sqrt{n!}}\,\langle n|\, i\varepsilon_0\left(\hat{a}e^{i(kz-\omega t)}-\hat{a}^\dagger e^{-i(kz-\omega t)}\right)\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\,|m\rangle \\
&= i\varepsilon_0 e^{-|\alpha|^2}\sum_{n,m=0}^{\infty}\frac{(\alpha^*)^n\,\alpha^m}{\sqrt{n!}\sqrt{m!}}\left(e^{i(kz-\omega t)}\,\langle n|\,\hat{a}\,|m\rangle - e^{-i(kz-\omega t)}\,\langle n|\,\hat{a}^\dagger\,|m\rangle\right) \\
&= i\varepsilon_0 e^{-|\alpha|^2}\sum_{n,m=0}^{\infty}\frac{(\alpha^*)^n\,\alpha^m}{\sqrt{n!}\sqrt{m!}}\left(e^{i(kz-\omega t)}\sqrt{n+1}\,\langle n+1\,|\,m\rangle - e^{-i(kz-\omega t)}\sqrt{m+1}\,\langle n\,|\,m+1\rangle\right) \\
&= i\varepsilon_0 e^{-|\alpha|^2}\left(e^{i(kz-\omega t)}\sum_{n=0}^{\infty}\frac{(\alpha^*)^n\,\alpha^{2n+1}}{\sqrt{n!}\sqrt{(n+1)!}}\sqrt{n+1} - e^{-i(kz-\omega t)}\sum_{m=0}^{\infty}\frac{(\alpha^*)^{m+1}\,\alpha^m}{\sqrt{(m+1)!}\sqrt{n!}}\sqrt{m+1}\right) \\
&= i\varepsilon_0 e^{-|\alpha|^2}\left(\alpha e^{i(kz-\omega t)} - \alpha^* e^{-i(kz-\omega t)}\right)\sum_{n=0}^{\infty}\frac{\left(|\alpha|^2\right)^n}{n!} \\
&= i\varepsilon_0 e^{-|\alpha|^2}\left(|\alpha|e^{i\phi}e^{i(kz-\omega t)} - |\alpha|e^{-i\phi}e^{-i(kz-\omega t)}\right)e^{|\alpha|^2} \\
&= i|\alpha|\varepsilon_0\left(e^{i(kz-\omega t+\phi)} - e^{-i(kz-\omega t+\phi)}\right) \\
&= -2|\alpha|\varepsilon_0\,\frac{e^{i(kz-\omega t+\phi)} - e^{-i(kz-\omega t+\phi)}}{2i}
\end{aligned}$$

$$\langle\alpha|\,\hat{E}(z,t)\,|\alpha\rangle = -2|\alpha|\varepsilon_0\sin(kz-\omega t+\phi)$$

The significance of this result is that the expectation value of the electric field operator of a single mode cavity in a coherent state is the same as what would be calculated in classical optics.

(h)

$$\begin{aligned}
\bar{n} &= \sum_n n\,P_n \\
&= \sum_{n=1}^{\infty} n\,|\langle n\,|\,\alpha\rangle|^2
\end{aligned}$$

$$\begin{aligned}
\langle n\,|\,\alpha\rangle &= \langle n|\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\,|m\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\,\langle n\,|\,m\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\frac{\alpha^n}{\sqrt{n!}} \\
\implies\ |\langle n\,|\,\alpha\rangle|^2 &= e^{-|\alpha|^2}\,\frac{\alpha^{2n}}{n!} \\
&= \frac{\lambda^n e^{-\lambda}}{n!},\ \lambda = |\alpha|^2
\end{aligned}$$

This is a Poisson distribution with parameter $|\alpha|^2$. Thus the average number of photons is simply the mean of this distribution, namely $\bar{n} = |\alpha|^2$, and the probability of finding n photons in a coherent state $|\alpha\rangle$ is simply $P_n = |\langle n\,|\,\alpha\rangle|^2 = \frac{\left(|\alpha|^2\right)^n e^{-|\alpha|^2}}{n!}$.

(i)

$$
\begin{aligned}
\langle \psi \,|\, \psi \rangle &= \frac{1}{\sqrt{2}} \left(\langle \beta | + \langle -\beta | \right) \frac{1}{\sqrt{2}} \left(|\beta\rangle + |-\beta\rangle \right) \\
&= \frac{1}{2} \left(\langle \beta \,|\, \beta \rangle + \langle \beta \,|\, -\beta \rangle + \langle -\beta \,|\, \beta \rangle + \langle -\beta \,|\, -\beta \rangle \right)
\end{aligned}
$$

$$
\begin{aligned}
\langle \alpha \,|\, \gamma \rangle &= \exp\!\left(-\frac{|\alpha|^2}{2} \right) \sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}} \langle n | \exp\!\left(-\frac{|\gamma|^2}{2} \right) \sum_{m=0}^{\infty} \frac{\gamma^n}{\sqrt{m!}} |m\rangle \\
&= \exp\!\left(-\frac{|\alpha|^2 + |\gamma|^2}{2} \right) \sum_{n,m=0}^{\infty} \frac{(\alpha^*)^n \, \gamma^m}{\sqrt{n!}\sqrt{m!}} \langle n \,|\, m \rangle \\
&= \exp\!\left(-\frac{|\alpha|^2 + |\gamma|^2}{2} \right) \sum_{n=0}^{\infty} \frac{(\alpha^* \gamma)^n}{n!}
\end{aligned}
$$

$$
\begin{aligned}
\langle \beta \,|\, \beta \rangle &= \exp\!\left(-\frac{|\beta|^2 + |\beta|^2}{2} \right) \sum_{n=0}^{\infty} \frac{(\beta^* \beta)^n}{n!} \\
&= e^{-|\beta|^2} \sum_{n=0}^{\infty} \frac{\left(|\beta|^2\right)^n}{n!} \\
&= e^{-|\beta|^2} e^{|\beta|^2} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\langle -\beta \,|\, \beta \rangle &= \exp\!\left(-\frac{|-\beta|^2 + |\beta|^2}{2} \right) \sum_{n=0}^{\infty} \frac{(-\beta^* \beta)^n}{n!} \\
&= e^{-|\beta|^2} \sum_{n=0}^{\infty} \frac{\left(-|\beta|^2\right)^n}{n!} \\
&= e^{-|\beta|^2} e^{-|\beta|^2} \\
&= e^{-2|\beta|^2}
\end{aligned}
$$

$$
\begin{aligned}
\langle \beta \,|\, -\beta \rangle &= \exp\!\left(-\frac{|\beta|^2 + |-\beta|^2}{2} \right) \sum_{n=0}^{\infty} \frac{(\beta^* (-\beta))^n}{n!} \\
&= e^{-|\beta|^2} \sum_{n=0}^{\infty} \frac{\left(-|\beta|^2\right)^n}{n!} \\
&= e^{-|\beta|^2} e^{-|\beta|^2} \\
&= e^{-2|\beta|^2}
\end{aligned}
$$

$$
\begin{aligned}
\langle -\beta \,|\, -\beta \rangle &= \exp\!\left(-\frac{|-\beta|^2 + |-\beta|^2}{2} \right) \sum_{n=0}^{\infty} \frac{(-\beta^* (-\beta))^n}{n!} \\
&= e^{-|\beta|^2} \sum_{n=0}^{\infty} \frac{\left(|\beta|^2\right)^n}{n!} \\
&= e^{-|\beta|^2} e^{|\beta|^2} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\implies \langle \psi \,|\, \psi \rangle &= \frac{1}{2} \left(1 + e^{-2|\beta|^2} + e^{-2|\beta|^2} + 1 \right) \\
&= 1 + e^{-2|\beta|^2} \\
&\approx 1 \text{ for } |\beta|^2 \gg 1
\end{aligned}
$$

Thus $|\psi\rangle$ is normalised for $|\beta|^2 \gg 1$.

$$
\begin{aligned}
\langle n \,|\, \psi \rangle &= \langle n | \frac{1}{\sqrt{2}} \left(|\beta\rangle + |-\beta\rangle \right) \\
&= \frac{1}{\sqrt{2}} \left(\langle n \,|\, \beta \rangle + \langle n \,|\, -\beta \rangle \right)
\end{aligned}
$$

$$
\text{(h)} \implies \langle n \,|\, \alpha \rangle = \exp\!\left(-\frac{|\alpha|^2}{2} \right) \frac{\alpha^n}{\sqrt{n!}}
$$

$$\langle n \,|\, \beta \rangle = \exp\left(-\frac{|\beta|^2}{2}\right) \frac{\beta^n}{\sqrt{n!}} \qquad\qquad \langle n \,|\, -\beta \rangle = \exp\left(-\frac{|\beta|^2}{2}\right) \frac{(-\beta)^n}{\sqrt{n!}}$$

$$\begin{aligned}
\implies \langle n \,|\, \psi \rangle &= \frac{1}{\sqrt{2}} \left[\exp\left(-\frac{|\beta|^2}{2}\right) \frac{\beta^n}{\sqrt{n!}} + \exp\left(-\frac{|\beta|^2}{2}\right) \frac{(-\beta)^n}{\sqrt{n!}}\right] \\
&= \frac{1}{\sqrt{2}\sqrt{n!}} \exp\left(-\frac{|\beta|^2}{2}\right) [\beta^n + (-\beta)^n] \\
&= \begin{cases} \dfrac{1}{\sqrt{2}\sqrt{n!}} \exp\left(-\dfrac{|\beta|^2}{2}\right) (\beta^n + \beta^n)\,, & n \text{ even} \\[2ex] \dfrac{1}{\sqrt{2}\sqrt{n!}} \exp\left(-\dfrac{|\beta|^2}{2}\right) (\beta^n - \beta^n)\,, & n \text{ odd} \end{cases} \\
&= \begin{cases} \sqrt{\dfrac{2}{n!}} \exp\left(-\dfrac{|\beta|^2}{2}\right) \beta^n, & n \text{ even} \\[2ex] 0, & n \text{ odd} \end{cases} \\
P_n &= |\langle n \,|\, \psi \rangle|^2 \\
&= \begin{cases} \dfrac{2e^{-|\beta|^2}\beta^{2n}}{n!}, & n \text{ even} \\[2ex] 0, & n \text{ odd} \end{cases}
\end{aligned}$$

(j)

$$\hat{H}_{\text{eff}} = \hbar\chi \left(\mathbb{1} \otimes |e\rangle \langle e| + \hat{a}^\dagger \hat{a} \otimes \hat{\sigma}_z\right), \quad |\psi(0)\rangle = |\alpha\rangle \otimes \frac{1}{\sqrt{2}} \left(|g\rangle + e^{i\phi} |e\rangle\right), \quad |\psi(t)\rangle = \exp\left(-\frac{i\hat{H}_{\text{eff}} t}{\hbar}\right) |\psi(0)\rangle$$

$$\begin{aligned}
|\psi(t)\rangle &= \exp\left(-\frac{i\hat{H}_{\text{eff}} t}{\hbar}\right) |\alpha\rangle \otimes \frac{1}{\sqrt{2}} \left(|g\rangle + e^{i\phi} |e\rangle\right) \\
&= \frac{1}{\sqrt{2}} \left[\exp\left(-\frac{i\hat{H}_{\text{eff}} t}{\hbar}\right) |\alpha\rangle \otimes |g\rangle + e^{i\phi} \exp\left(-\frac{i\hat{H}_{\text{eff}} t}{\hbar}\right) |\alpha\rangle \otimes |e\rangle\right] \\
&= \frac{1}{\sqrt{2}} \left(|\psi_{\alpha g}(t)\rangle + e^{i\phi} |\psi_{\alpha e}(t)\rangle\right)
\end{aligned}$$

To see how $|\psi(t)\rangle$ evolves in time, we must determine how $|\psi_{\alpha g}(t)\rangle = |\alpha\rangle \otimes |g\rangle$ and $|\psi_{\alpha e}(t)\rangle = |\alpha\rangle \otimes |e\rangle$ evolve in time. Let us first consider how $|\psi_{ng}(0)\rangle = |n\rangle \otimes |g\rangle$ and $|\psi_{ne}(0)\rangle = |n\rangle \otimes |e\rangle$ evolve.

$$\begin{aligned}
|\psi_{ng}(t)\rangle &= \exp\left(-\frac{i\hat{H}_{\text{eff}}t}{\hbar}\right)|\psi_{ng}(0)\rangle \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i\hat{H}_{\text{eff}}t}{\hbar}\right)^k |n\rangle \otimes |g\rangle \\
&= \sum_{k=0}^{\infty}\frac{(-i\chi t)^k}{k!}\left(\mathbb{1}\otimes|e\rangle\langle e| + \hat{a}^\dagger\hat{a}\otimes\hat{\sigma}_z\right)^k |n\rangle \otimes |g\rangle \\
&= \sum_{k=0}^{\infty}\frac{(-i\chi t)^k}{k!}(0 + n(-1))^k |n\rangle \otimes |g\rangle \\
&= \sum_{k=0}^{\infty}\frac{(i\chi n t)^k}{k!}|n\rangle \otimes |g\rangle \\
&= e^{i\chi n t}|n\rangle \otimes |g\rangle
\end{aligned}$$

$$\begin{aligned}
|\psi_{ne}(t)\rangle &= \exp\left(-\frac{i\hat{H}_{\text{eff}}t}{\hbar}\right)|\psi_{ne}(0)\rangle \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i\hat{H}_{\text{eff}}t}{\hbar}\right)^k |n\rangle \otimes |e\rangle \\
&= \sum_{k=0}^{\infty}\frac{(-i\chi t)^k}{k!}\left(\mathbb{1}\otimes|e\rangle\langle e| + \hat{a}^\dagger\hat{a}\otimes\hat{\sigma}_z\right)^k |n\rangle \otimes |e\rangle \\
&= \sum_{k=0}^{\infty}\frac{(-i\chi t)^k}{k!}(1 + n)^k |n\rangle \otimes |e\rangle \\
&= \sum_{k=0}^{\infty}\frac{(-i\chi (n+1) t)^k}{k!}|n\rangle \otimes |e\rangle \\
&= e^{-i\chi (n+1) t}|n\rangle \otimes |e\rangle
\end{aligned}$$

Now let us see how $|\psi_{\alpha g}(t)\rangle = |\alpha\rangle \otimes |g\rangle$ and $|\psi_{\alpha e}(t)\rangle = |\alpha\rangle \otimes |e\rangle$ evolve in time.

$$\begin{aligned}
|\psi_{\alpha g}(t)\rangle &= \exp\left(-\frac{i\hat{H}_{\text{eff}}t}{\hbar}\right)\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \otimes |g\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp\left(-\frac{i\hat{H}_{\text{eff}}t}{\hbar}\right)|\psi_{ng}(0)\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|\psi_{ng}(t)\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{i\chi n t}|n\rangle \otimes |g\rangle \\
&= \exp\left(-\frac{\left|\alpha e^{i\chi t}\right|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\left(\alpha e^{i\chi t}\right)^n}{\sqrt{n!}}|n\rangle \otimes |g\rangle \\
&= \left|\alpha e^{i\chi t}\right\rangle \otimes |g\rangle
\end{aligned}$$

$$\begin{aligned}
|\psi_{\alpha e}(t)\rangle &= \exp\left(-\frac{i\hat{H}_{\text{eff}}t}{\hbar}\right)\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \otimes |e\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp\left(-\frac{i\hat{H}_{\text{eff}}t}{\hbar}\right)|\psi_{ne}(0)\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|\psi_{ne}(t)\rangle \\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\chi (n+1) t}|n\rangle \otimes |e\rangle \\
&= e^{-i\chi t}\exp\left(-\frac{\left|\alpha e^{-i\chi t}\right|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-i\chi t}\right)^n}{\sqrt{n!}}|n\rangle \otimes |e\rangle \\
&= e^{-i\chi t}\left|\alpha e^{-i\chi t}\right\rangle \otimes |e\rangle
\end{aligned}$$

We can now determine $|\psi(t)\rangle$ using $|\psi_{\alpha g}(t)\rangle$ and $|\psi_{\alpha e}(t)\rangle$.

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\left|\alpha e^{i\chi t}\right\rangle \otimes |g\rangle + e^{i(\phi-\chi t)}\left|\alpha e^{-i\chi t}\right\rangle \otimes |e\rangle\right)$$

$$\begin{aligned}
\left|\psi\left(\frac{\pi}{2\chi}\right)\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|\alpha e^{\frac{i\pi}{2}}\right\rangle \otimes |g\rangle + e^{i\left(\phi-\frac{\pi}{2}\right)}\left|\alpha e^{-\frac{i\pi}{2}}\right\rangle \otimes |e\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|i\alpha\rangle \otimes |g\rangle - ie^{i\phi}|-i\alpha\rangle \otimes |e\rangle\right)
\end{aligned}$$

The state vector for $\chi t = \frac{\pi}{2}$ is analogous to the state vector for Schrödinger's cat, where the coherent field states act as the state of the cat.

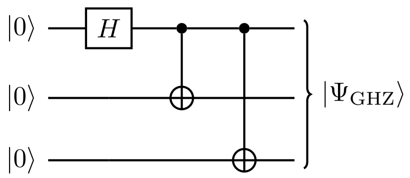
(k)

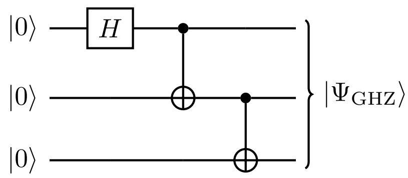
$$
\begin{aligned}
& && |000\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \\
&\text{Apply } \hat{\mathrm{H}} \otimes \mathbb{1} \otimes \mathbb{1} && \to \hat{\mathrm{H}} |0\rangle \otimes |0\rangle \otimes |0\rangle \\
& && = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |0\rangle \otimes |0\rangle \\
&\text{Apply } \left(|0\rangle \langle 0| \otimes \mathbb{1} + |1\rangle \langle 1| \otimes \hat{\mathrm{X}}\right) \otimes \mathbb{1} && \to \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \otimes |0\rangle \\
&\text{Apply } |0\rangle \langle 0| \otimes \mathbb{1} \otimes \mathbb{1} + |1\rangle \langle 1| \otimes \mathbb{1} \otimes \hat{\mathrm{X}} && \to \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle) \\
& && = |\Psi_{\mathrm{GHZ}}\rangle
\end{aligned}
$$

Alternatively, the last C-NOT gate can be changed so that the control qubit is the second qubit as opposed to the first:

$$
\begin{aligned}
& && \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \otimes |0\rangle \\
&\text{Apply } \mathbb{1} \otimes \left(|0\rangle \langle 0| \otimes \mathbb{1} + |1\rangle \langle 1| \otimes \hat{\mathrm{X}}\right) && \to \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle) \\
& && = |\Psi_{\mathrm{GHZ}}\rangle
\end{aligned}
$$

Thus the following two circuit diagrams both transform $|000\rangle$ into $|\Psi_{\mathrm{GHZ}}\rangle$:





$$
\begin{aligned}
|\Psi_{\mathrm{GHZ}}\rangle \langle \Psi_{\mathrm{GHZ}}| &= \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \frac{1}{\sqrt{2}} (\langle 000| + \langle 111|) \\
&= \frac{1}{2} (|000\rangle \langle 000| + |000\rangle \langle 111| + |111\rangle \langle 000| + |111\rangle \langle 111|)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho} &= \mathrm{tr}_3(|\Psi_{\mathrm{GHZ}}\rangle \langle \Psi_{\mathrm{GHZ}}|) \\
&= \frac{1}{2} \mathrm{tr}(|000\rangle \langle 000| + |000\rangle \langle 111| + |111\rangle \langle 000| + |111\rangle \langle 111|) \\
\hat{\rho} &= \frac{1}{2} (|00\rangle \langle 00| + |11\rangle \langle 11|) \\
&= \frac{1}{2} (|0\rangle \langle 0| \otimes |0\rangle \langle 0| + |1\rangle \langle 1| \otimes |1\rangle \langle 1|)
\end{aligned}
$$

As the reduced state is separable, it is not entangled.

2 Numerical Exercises

2.1 Abstract

In this report, the XXZ Heisenberg model is studied. The time evolution of the entanglement entropy and quantum Fisher information of the system is calculated, and the corresponding relationship between these quantities is determined. An explanation of how to perform an equivalent experiment on a quantum computer is also included, before concluding with a brief discussion of the limitations of the report.

2.2 Background

2.2.1 Density Operators, Entropy, & Time Evolution

For a given ensemble state $|\Psi\rangle$, the corresponding density operator $\hat{\rho}$ is given by [1]

$$\hat{\rho} = |\Psi\rangle \langle\Psi| .$$

If $|\Psi\rangle$ can be written as a composition of states $|\Psi_a\rangle |\Psi_b\rangle$, then the reduced density operator $\hat{\rho}_a$ is given by

$$\hat{\rho}_a = \mathrm{tr}_b(\hat{\rho}) = \mathrm{tr}(|\Psi_b\rangle \langle\Psi_b|) |\Psi_a\rangle \langle\Psi_a| ,$$

and analogously for $\hat{\rho}_b$.

The von Neumann entropy $S_{\mathrm{vN}}(\hat{\rho})$ of a given density operator $\hat{\rho}$ with eigenvalues $\{\lambda_i\}$ is given by

$$S_{\mathrm{vN}}(\hat{\rho}) = -\,\mathrm{tr}(\hat{\rho} \ln \hat{\rho}) = -\sum_i \lambda_i \ln \lambda_i, \tag{1}$$

where $\ln A = B \iff A = e^B$. The von Neumann entropy of a density operator corresponding to a pure state ($\hat{\rho}^2 = \hat{\rho}$) is 0. For mixed states, S_{vN} is greater than 0, and thus can be considered as a determination of how mixed a state is. For high-dimensional systems, the latter expression of Equation 1 involving the operator's eigenvalues is often far easier to compute than the former involving a matrix logarithm.

The entanglement entropy $S_{\mathrm{ent}}(\hat{\rho})$ for a system of two subsystems a and b is defined as the von Neumann entropy of either of the reduced density operators $\hat{\rho}_a$, $\hat{\rho}_b$, i.e. [2]

$$S_{\mathrm{ent}}(\hat{\rho}) = S_{\mathrm{vN}}(\hat{\rho}_a) = S_{\mathrm{vN}}(\hat{\rho}_b) , \tag{2}$$

and is a measure of the degree of entanglement between the two subsystems.

In the Schrödinger picture of quantum mechanics, a state vector $|\Psi\rangle = |\Psi(0)\rangle$ evolves in time via

$$|\Psi(t)\rangle = U(t) |\Psi(0)\rangle = \exp\left(-\frac{i\hat{H}t}{\hbar}\right) |\Psi(0)\rangle , \tag{3}$$

where $\hat{H}$ is the (time-independent) Hamiltonian of the system, and $U(t)$ is the time-evolution operator.

2.2.2 Quantum Heisenberg Model

The quantum Heisenberg model [3] is a 1-dimensional quantum spin chain consisting of a number of spin-$\frac{1}{2}$ particles that interact with their nearest neighbours. The Hamiltonian of a system of N particles is given by

$$\hat{H} = -\sum_{\alpha=1}^{3} \sum_{j=1}^{N-1} J_\alpha \hat{\sigma}_j^\alpha \hat{\sigma}_{j+1}^\alpha,$$

$$\hat{O}_j \equiv \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \underbrace{\hat{O}}_{j\text{th position}} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1},$$

where J_α are coupling constants and σ^α are the Pauli matrices

$$\sigma^1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This report focuses on the XXZ model, i.e. $J_x = J_y \equiv J$, $J_z \equiv -U$:

$$\hat{H} = -J \sum_{j=1}^{N-1} \left(\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y \right) + U \sum_{j=1}^{N-1} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z. \tag{4}$$

The system of N spin sites can be divided into two susbytems, where subsystem i consists of a chain of N_i spin sites. The quantum Fisher information for the XXZ model at time t is given by [4]

$$\begin{aligned} F_{\mathrm{Q}}(t) &= \sum_{j,k=1}^{N} s_j s_k \left\langle \Psi(t) \,\middle|\, \hat{\sigma}_j^z \hat{\sigma}_k^z \,\middle|\, \Psi(t) \right\rangle - \left(\sum_{j=1}^{N} s_j \left\langle \Psi(t) \,\middle|\, \hat{\sigma}_j^z \,\middle|\, \Psi(t) \right\rangle \right)^2 \\ &= N + 2 \sum_{j<k} s_j s_k \left\langle \Psi(t) \,\middle|\, \hat{\sigma}_j^z \hat{\sigma}_k^z \,\middle|\, \Psi(t) \right\rangle - \left(\sum_{j=1}^{N} s_j \left\langle \Psi(t) \,\middle|\, \hat{\sigma}_j^z \,\middle|\, \Psi(t) \right\rangle \right)^2, \\ s_j &= \begin{cases} 1, & j \leqslant N_1 \\ -1, & j > N1 \end{cases}. \end{aligned} \tag{5}$$

The latter expression is arrived at by noticing that $s_j s_j = 1$, $\hat{\sigma}_j^z \hat{\sigma}_j^z = \mathbb{1}$, and $\langle \Psi(t) \,|\, \Psi(t) \rangle = 1$, and that the term in the first sum is invariant under the change $j \leftrightarrow k$. For the XX model ($U = 0$), the quantum Fisher information can be related to the entanglement entropy via [4]

$$S_{\mathrm{ent}} \approx \frac{5}{32} F_{\mathrm{Q}}. \tag{6}$$

This approximate relation holds regardless of how the subsytems are split.

2.3 Methodology

Defining $\hat{H}' = \frac{\hat{H}}{J}$ and $U' = \frac{U}{J}$, the XXZ Hamiltonian in Equation 4 can be rewritten as

$$\begin{aligned} \hat{H}' &= -\sum_{j=1}^{N-1} \left(\hat{\sigma}_j^x \otimes \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \otimes \hat{\sigma}_{j+1}^y - U' \hat{\sigma}_j^z \otimes \hat{\sigma}_{j+1}^z \right) \\ &= -\sum_{j=1}^{N-1} \hat{\Sigma}_j, \end{aligned} \tag{7}$$

where $\hat{\Sigma} \equiv \hat{\sigma}^x \otimes \hat{\sigma}^x + \hat{\sigma}^y \otimes \hat{\sigma}^y - U' \hat{\sigma}^z \otimes \hat{\sigma}^z$. The corresponding time evolution operator in Equation 3 can then be given as

$$U(t) \equiv \exp\left(-\frac{i\hat{H}t}{\hbar} \right) = \exp\left(it' \sum_{j=1}^{N-1} \hat{\Sigma}_j \right), \tag{8}$$

where $t' \equiv \frac{Jt}{\hbar}$. Equation 8 can then be used to evolve a state at time t over a short time Δt via

$$|\Psi(t + \Delta t)\rangle = U(\Delta t) \, |\Psi(t)\rangle \, . \tag{9}$$

Equation 9 can be used to evolve a state from $t = 0$ to $t = T$ by repeated application of $U(\Delta t)$ on $|\Psi(0)\rangle$. While it would be equivalent to instead generate $U(t_1), \ldots, U(t_{n-1}), U(t_n) = U(T)$ and act each of these on $|\Psi(0)\rangle$, it would be sufficiently more computationally expensive to calculate $n = \frac{T}{\Delta t}$ matrix exponentials.

Functions were written in Python to calculate $U(t)$ (Equation 8), $S_{\mathrm{ent}}(t) \equiv S_{\mathrm{ent}}(\hat{\rho}(t))$ (Equation 2), and $F_{\mathrm{Q}}(t)$ (Equation 5). $S_{\mathrm{ent}}(t)$ and $F_{\mathrm{Q}}(t)$ were calculated for $U = 0, J$ (i.e. $U' = 0, 1$), and the accuracy of the approximate relation in Equation 6 was tested for both cases. The initial state of the system in all cases was chosen as $|\Psi(0)\rangle = |\Psi_{\text{Néel}}\rangle = |{\uparrow\downarrow\uparrow\downarrow} \ldots {\uparrow\downarrow}\rangle$, i.e. an antiferromagnetic state.

2.4 Results & Discussion

2.4.1 Entanglement Entropy Equivalence

By the equality in Equation 2, calculating the entanglement entropy of a density operator does not depend on which reduced density operator is used. Figure 2 shows an example of this equivalence.

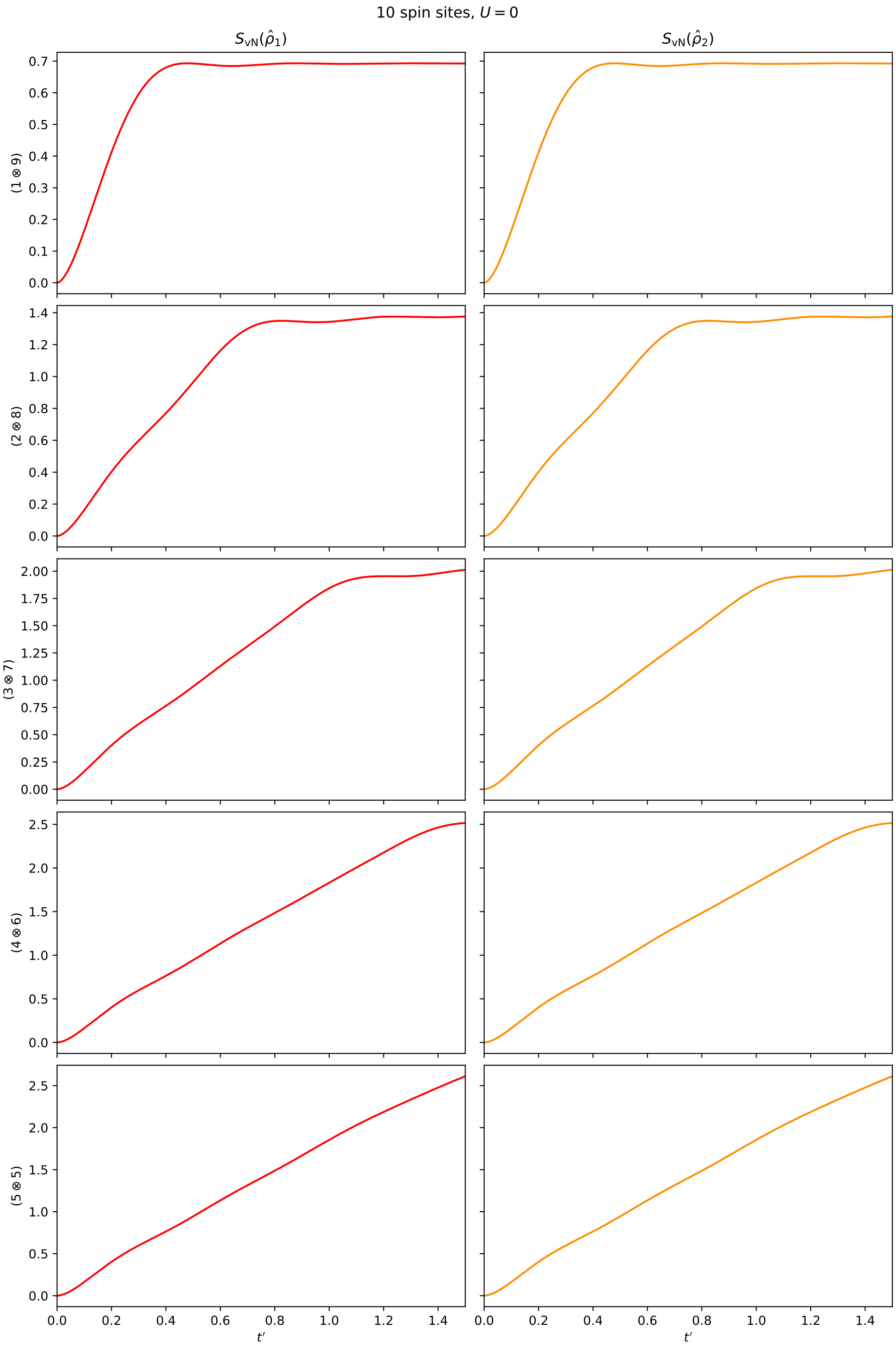
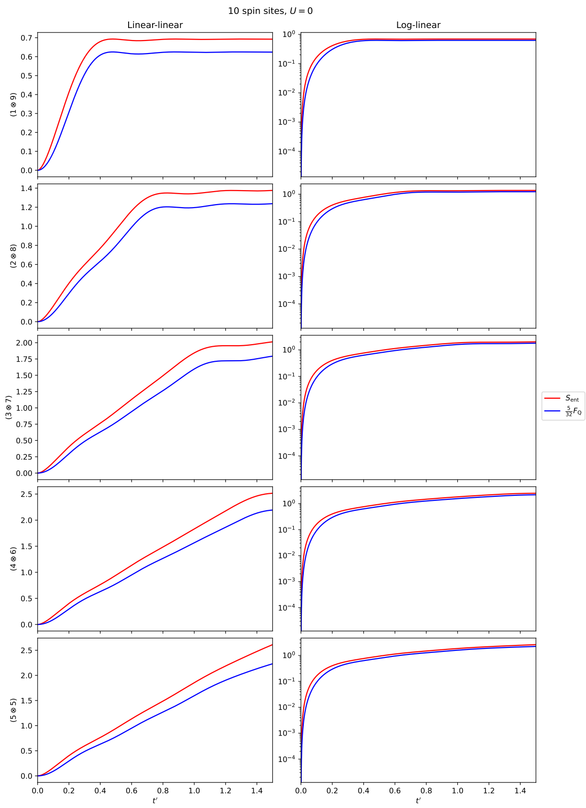


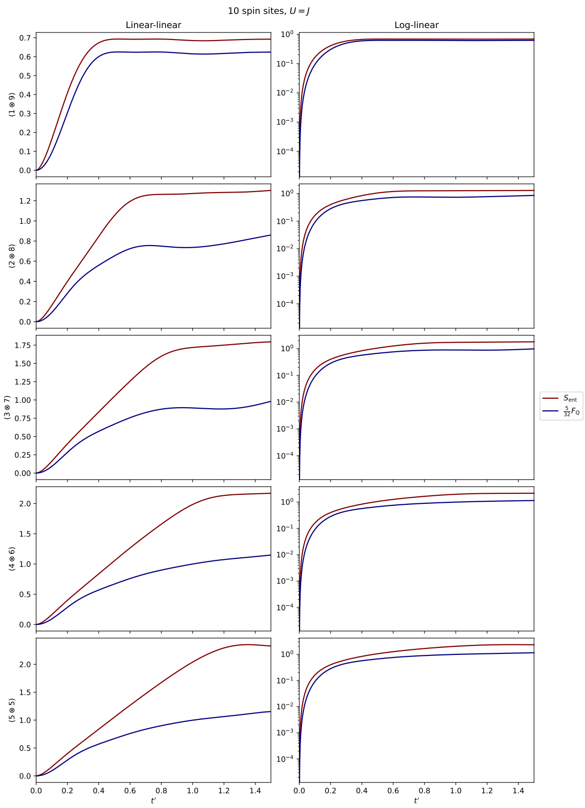
Figure 2: Von Neumann entropy of both reduced density operators for each set of subsystems of the $N = 10$ XX model.

2.4.2 Quantum Fisher Information vs. Entanglement Entropy

The time evolution of the state vector $|\Psi(t)\rangle$ was calculated up to $t' = 1.5$ for both the XX ($U = 0$) and $U = J$ XXZ models, for $N = 10$ spin sites. The corresponding entanglement entropy $S_{\text{ent}}(t)$ and quantum Fisher information $F_{\text{Q}}(t)$ for each possible set of subsystems ($N_1 = 1, 2, 3, 4, 5$) were calculated and plotted (Figure 3).



(a) XX model.



(b) XXZ model.

Figure 3: Entanglement entropy (red) and scaled quantum Fisher information (blue) for each set of subsystems of the $N = 10$ XX (left) and $U = J$ XXZ (right) models.

For the XX model, the relation in Equation 6 is a good approximation of the relation between the entanglement entropy and quantum Fisher information, as can be seen from Figure 3a. It can also be seen that increasing the difference between subsystem sizes corresponds to entropy and information having reaching a smaller maximum value in a shorter amount of time.

For the XXZ model, however, the relation in Equation 6 only seems to hold for small times or for the case of $N_1 = 1$, as can be seen from Figure 3b, and so is generally not a good approximation.

To obtain a better understanding of the accuracy of the relation between entanglement entropy and quantum Fisher information, the percentage error of Equation 6 was calculated by treating $\frac{5}{32}F_{\text{Q}}$ as an estimator of S_{ent}, i.e.

$$\%\ \text{error} = \frac{\left|S_{\text{ent}} - \frac{5}{32}F_{\text{Q}}\right|}{S_{\text{ent}}} = \frac{\left|\frac{S_{\text{ent}}}{F_{\text{Q}}} - \frac{5}{32}\right|}{\frac{S_{\text{ent}}}{F_{\text{Q}}}}. \tag{10}$$

The ratio of entanglement entropy and quantum Fisher information, and the corresponding percentage error (Equation 10), were calculated and plotted for the $N = 10$ XX and XXZ models (Figure 4).

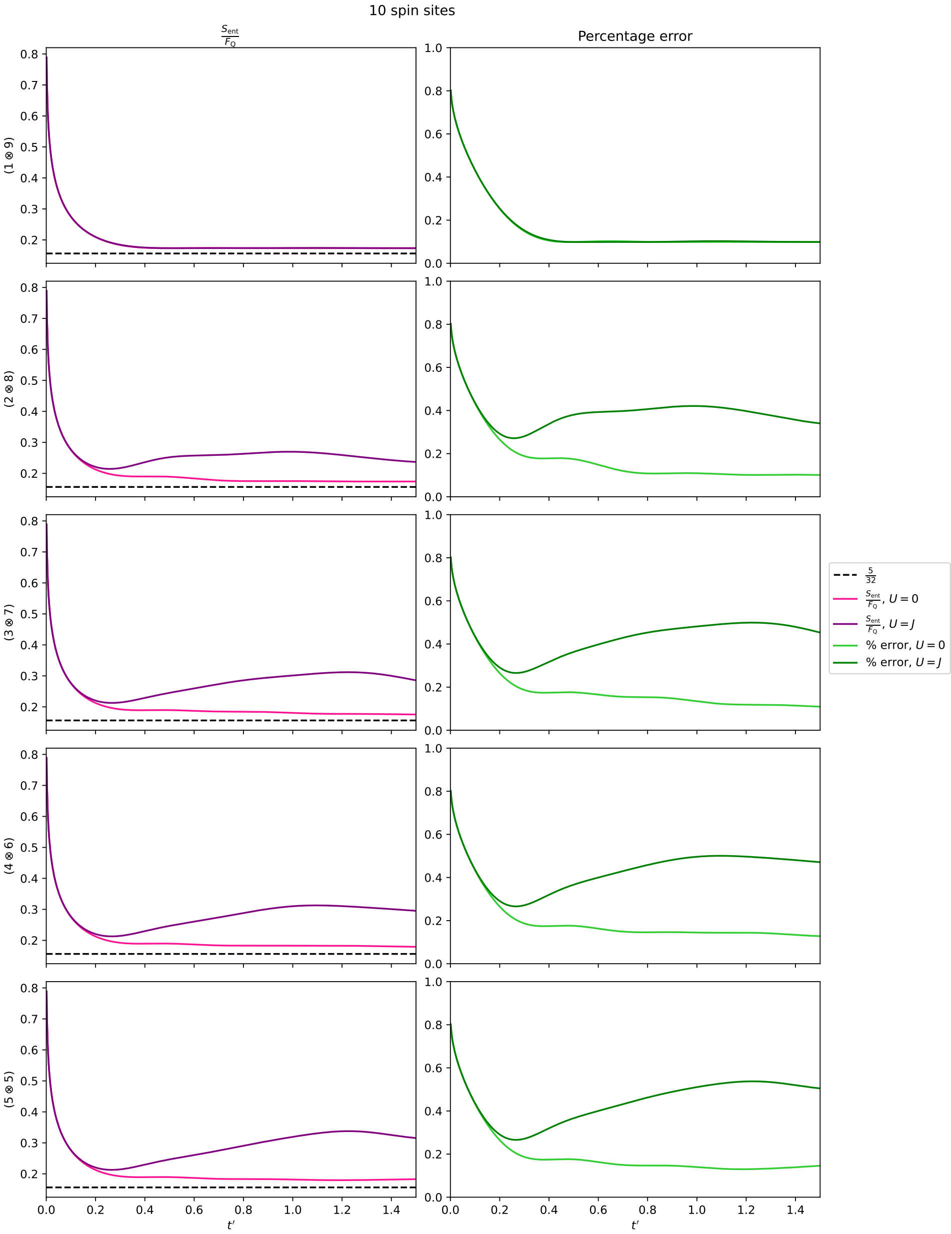
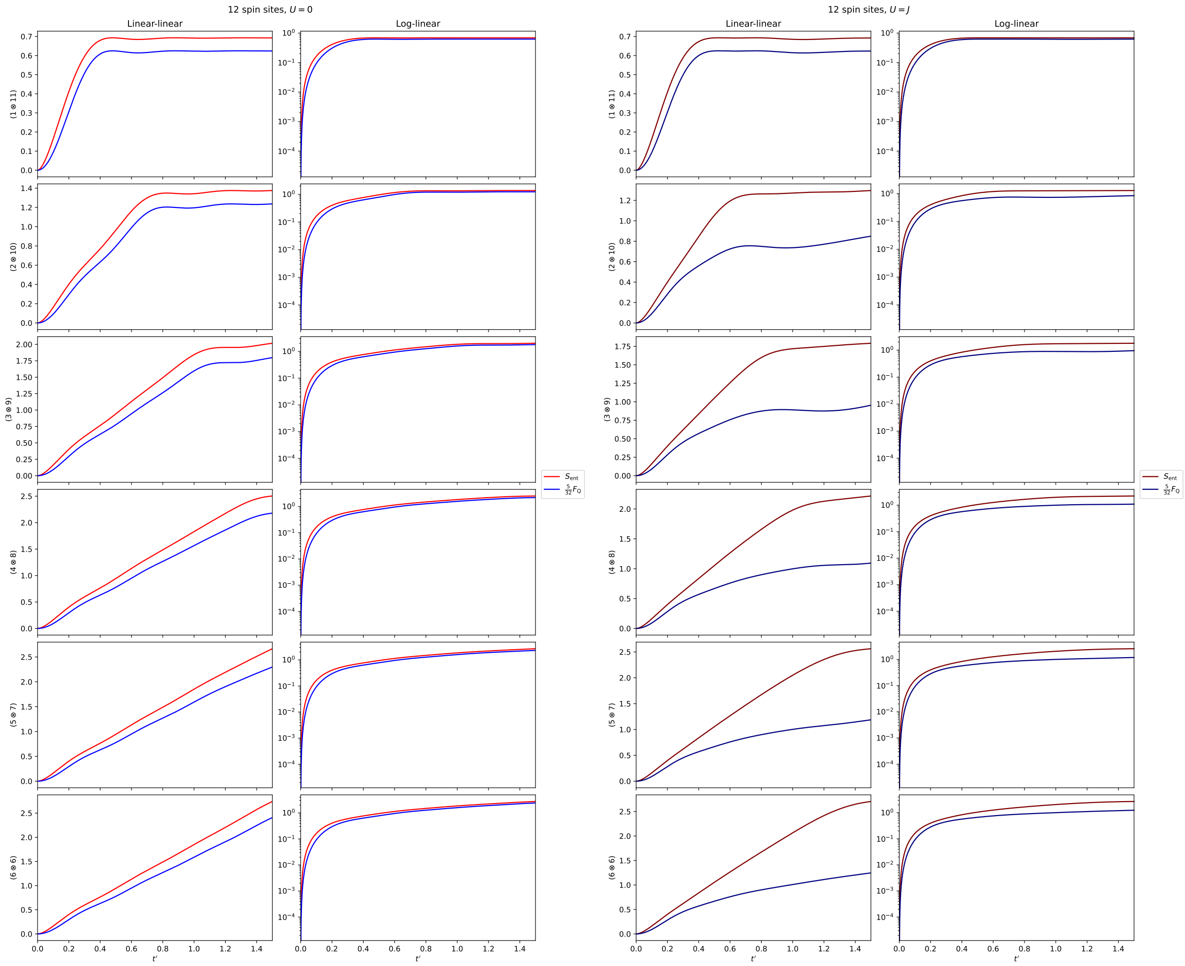


Figure 4: Ratio of entanglement entropy and quantum Fisher information (left) and corresponding percentage error (right) for each set of subsystems of the $N = 10$ XX (light) and $U = J$ XXZ (dark) models. The dashed line corresponds to the approximate ratio given by Equation 6.

It can be seen from Figure 4 that, for the XX model and for the XXZ model when $N_1 = 1$, the ratio of the entanglement entropy and quantum Fisher information appears to asymptote to $\frac{5}{32}$ as t' increases. The corresponding percentage error is also far lower for these cases, again showing when the relation in Equation 6 holds.

These calculations were repeated for $N = 12$, and analogous plots to Figure 3 and Figure 4 were made (Figure 5, Figure 6), again showing the same relationship. The results for $N = 10$ were further extended up to $t' = 10$ (Figure 7, Figure 8), where it was found that the entanglement entropy and quantum Fisher information, as well as their ratio and corresponding percentage error, exhibited unsteady behaviour for longer times.



(a) XX model.

(b) XXZ model.

Figure 5: Entanglement entropy (red) and scaled quantum Fisher information (blue) for each set of subsystems of the $N = 12$ XX (left) and $U = J$ XXZ (right) models.

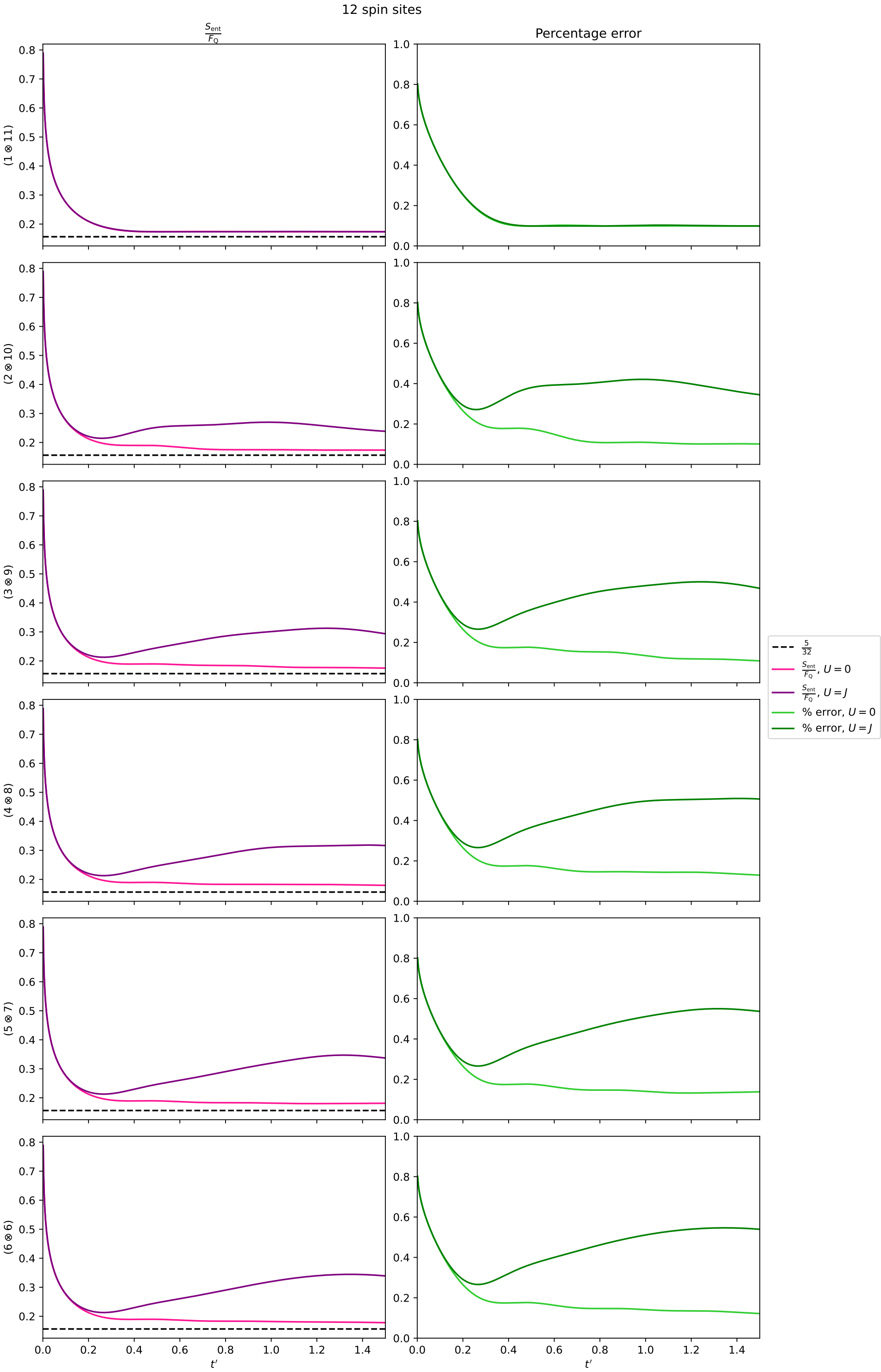
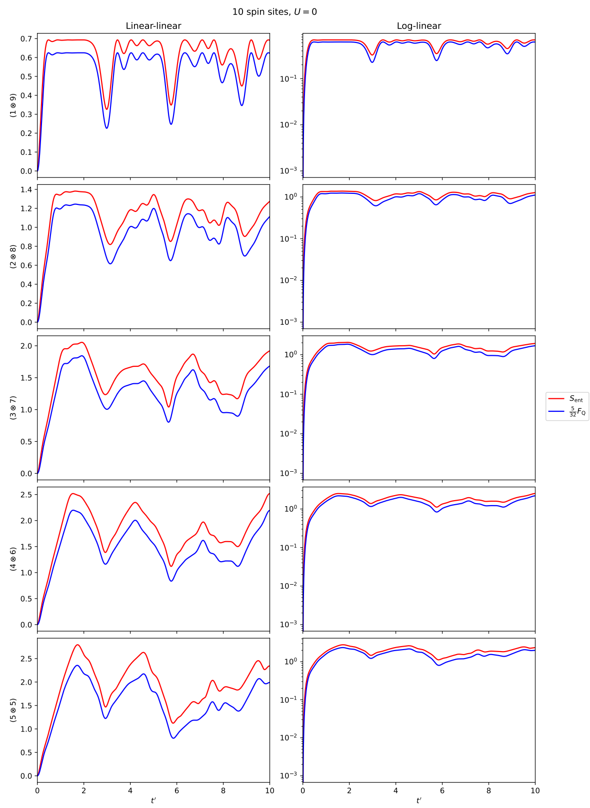
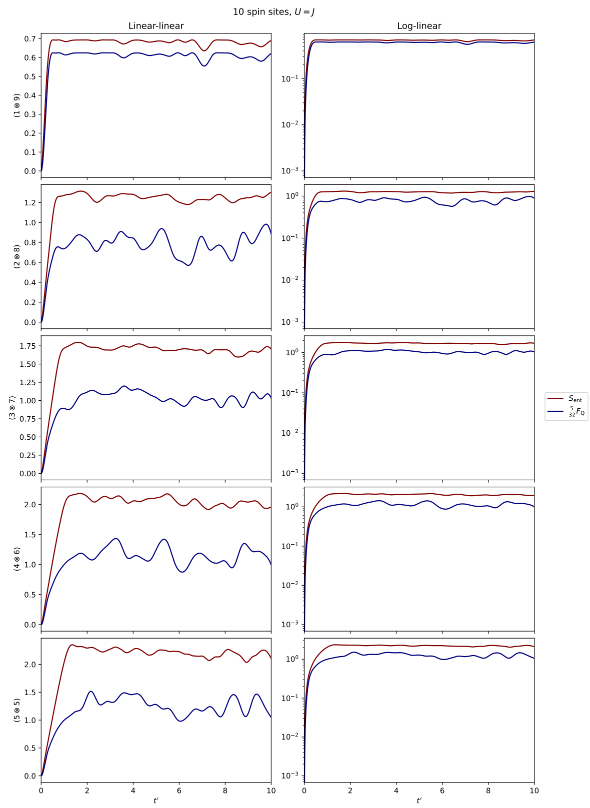


Figure 6: Ratio of entanglement entropy and quantum Fisher information (left) and corresponding percentage error (right) for each set of subsystems of the $N = 12$ XX (light) and $U = J$ XXZ (dark) models. The dashed line corresponds to the approximate ratio given by Equation 6.



(a) XX model.



(b) XXZ model.

Figure 7: Entanglement entropy (red) and scaled quantum Fisher information (blue) for each set of subsystems of the $N = 10$ XX (left) and $U = J$ XXZ (right) models.

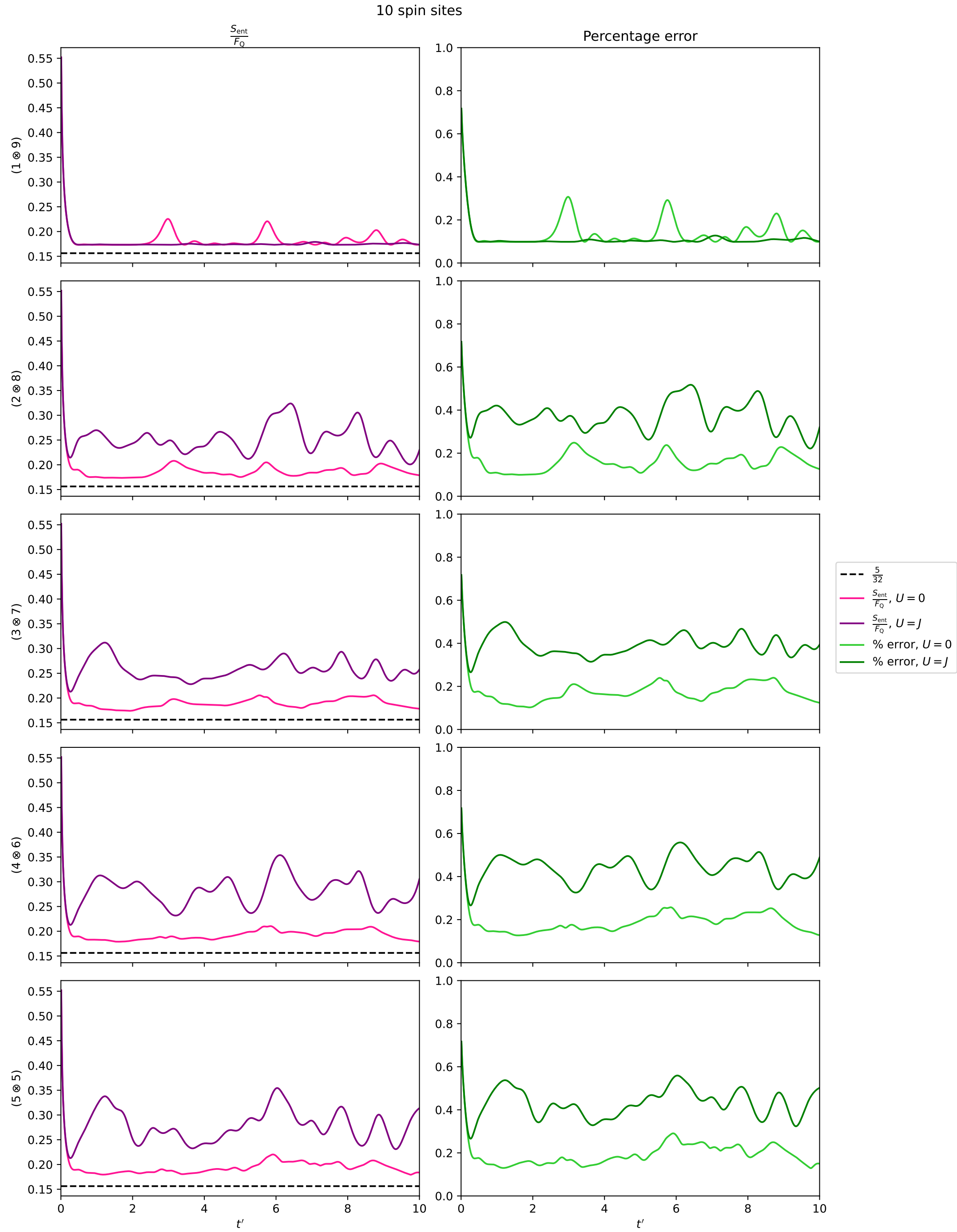


Figure 8: Ratio of entanglement entropy and quantum Fisher information (left) and corresponding percentage error (right) for each set of subsystems of the $N = 10$ XX (light) and $U = J$ XXZ (dark) models. The dashed line corresponds to the approximate ratio given by Equation 6.

2.4.3 Quantum Computer Simulation

The time evolution operator $U(t)$ for a Hamiltonian $\hat{H}$ can be written using the first-order Suzuki-Trotter decomposition via [5, 6]

$$\exp\left(-\frac{i\hat{H}t}{\hbar}\right) = \lim_{n\to\infty}\left[\prod_j \exp\left(-\frac{i\hat{H}_j t}{n\hbar}\right)\right]^n, \tag{11}$$

where $\hat{H} = \sum_j \hat{H}_j$. The Hamiltonian $\hat{H}'$ used in this report (Equation 7) can be split into two non-commuting parts; as we have $\left[\hat{\Sigma}_j, \hat{\Sigma}_k\right] = 0$ for all $|j-k| \neq 1$, Equation 7 can then be rewritten as

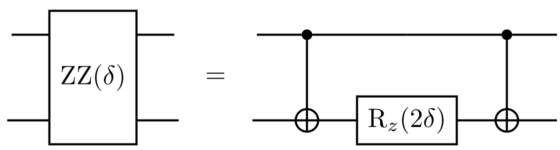
$$\hat{H}' = -\sum_{j=1}^{\frac{N}{2}-1} \hat{\Sigma}_{2j} - \sum_{j=1}^{\frac{N}{2}} \hat{\Sigma}_{2j-1} = \hat{H}'_{\text{even}} + \hat{H}'_{\text{odd}}. \tag{12}$$

Equation 11 and Equation 12 can therefore be combined to obtain an approximation for the time evolution operator for the XXZ model as

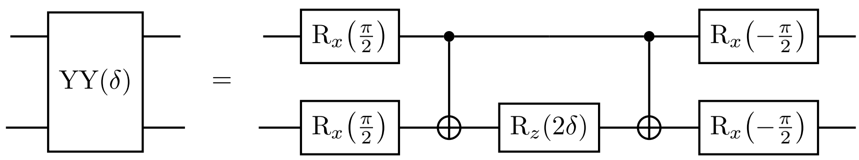
$$U(t) \approx \left[\exp\left(-\frac{i\hat{H}'_{\text{even}}t'}{n}\right)\exp\left(-\frac{i\hat{H}'_{\text{odd}}t'}{n}\right)\right]^n. \tag{13}$$

To realise this approximate time evolution operator as a quantum circuit, it is first necessary to define the XX, YY and ZZ gates as [7]

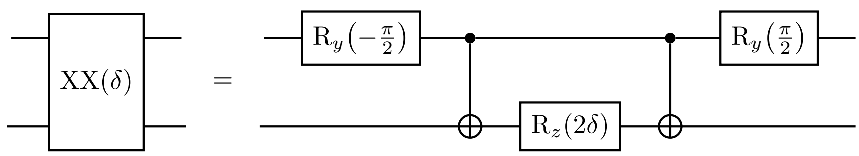
$\exp(-i\delta\,\hat{\sigma}^z \otimes \hat{\sigma}^z) \equiv$



$\exp(-i\delta\,\hat{\sigma}^y \otimes \hat{\sigma}^y) \equiv$



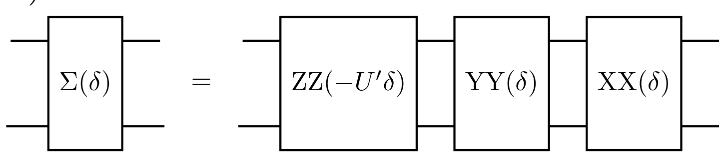
$\exp(-i\delta\,\hat{\sigma}^x \otimes \hat{\sigma}^x) \equiv$



These gates can then be combined to form a Σ gate as

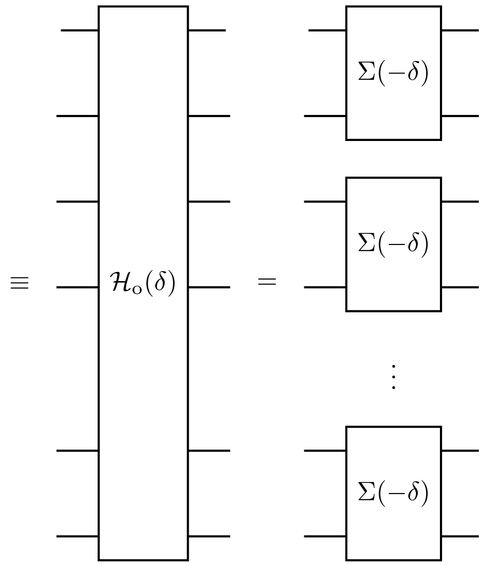
$$\exp\left(-i\delta\,\hat{\Sigma}\right) = \exp[-i\delta\,\hat{\sigma}^x \otimes \hat{\sigma}^x]\exp[-i\delta\,\hat{\sigma}^y \otimes \hat{\sigma}^y]\exp[-i\,(-U'\delta)\,\hat{\sigma}^z \otimes \hat{\sigma}^z]$$

$\equiv$

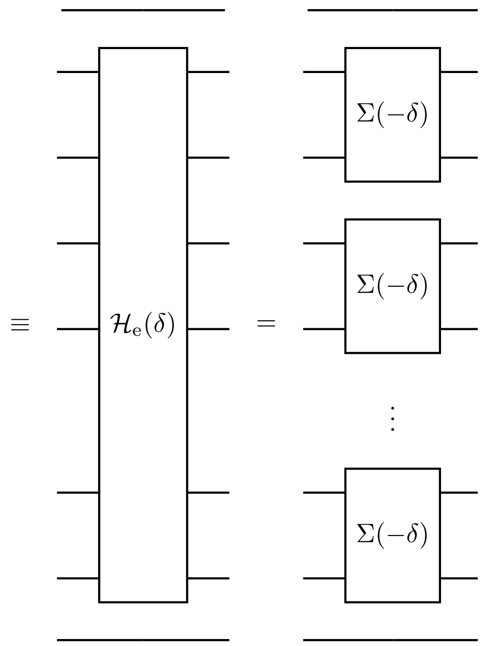


Combinations of Σ gates can be used to define $\mathcal{H}_\mathrm{o}$ and $\mathcal{H}_\mathrm{e}$ gates as

$$\exp\left(-i\delta\,\hat{H}'_{\mathrm{odd}}\right) = \exp\left[-i\,(-\delta)\,\hat{\Sigma}_1\right]\exp\left[-i\,(-\delta)\,\hat{\Sigma}_3\right]\cdots\exp\left[-i\,(-\delta)\,\hat{\Sigma}_{N-1}\right]$$

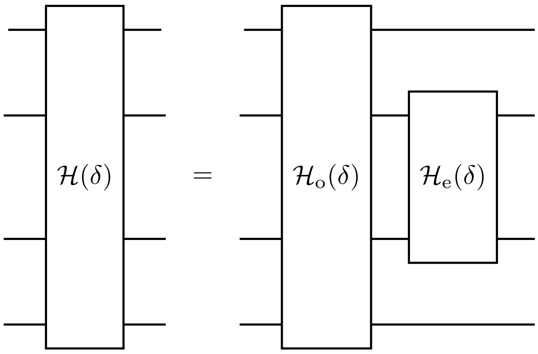


$$\exp\left(-i\delta\,\hat{H}'_{\mathrm{even}}\right) = \exp\left[-i\,(-\delta)\,\hat{\Sigma}_2\right]\exp\left[-i\,(-\delta)\,\hat{\Sigma}_4\right]\cdots\exp\left[-i\,(-\delta)\,\hat{\Sigma}_{N-2}\right]$$



The $\mathcal{H}_{\text{o,e}}$ gates can be used to define a $\mathcal{H}$ gate as

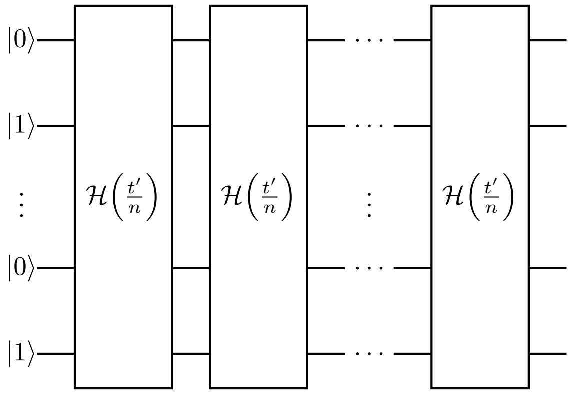
$$\exp\left(-i\delta\,\hat{H}'_{\text{even}}\right)\exp\left(-i\delta\,\hat{H}'_{\text{odd}}\right) \equiv$$



The $\mathcal{H}$ gate can then be used to approximate the time evolution of $|\Psi(0)\rangle = |\Psi_{\text{Néel}}\rangle$ via Equation 13 as

$$|\Psi(t)\rangle = \exp\left(-i\hat{H}'t'\right)|\Psi_{\text{Néel}}\rangle \approx \left[\exp\left(-i\frac{t'}{n}\hat{H}'_{\text{even}}\right)\exp\left(-i\frac{t'}{n}\hat{H}'_{\text{odd}}\right)\right]^n |\Psi_{\text{Néel}}\rangle$$

$$=$$



Quantum state tomography [8] can be employed to obtain the density matrix of a given state via repeated measurements, where the entanglement entropy can then be calculated as before. As each term in the quantum Fisher information (Equation 5) is an expected value of an operator, it can be calculated via ensemble averages over repeated measurements of the system. This first-order "Trotterisation" has error of the order $\mathcal{O}\left(\frac{(t')^2}{n}\right)$, [7] and is thus most accurate for shorter times t' and larger number of time slices n.

2.5 Conclusion

The relation between entanglement entropy and quantum Fisher information in Equation 6 was found to only hold for the XX model, or in the outlier case of the XXZ model when one subsystem has a single spin site. It was also shown that, as expected, calculating the entanglement entropy does not depend on which reduced density operator is used.

The method used in this report to calculate the time evolution operator involved directly calculating and exponentiating the Hamiltonian matrix. As the size of this matrix is $2^N \times 2^N$, where each component is a 128 bit complex number, this method very quickly takes extraordinary amounts of time and computational power to run; indeed, attempting to repeat the calculations in this report for $N = 14$ required over 21GB of memory. Rewriting the programme to instead use sparse matrix exponentiation still required over 4GB of memory.

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