PYU44T20 Quantum Optics and Information
Problem Set 2 due 23/03/2023

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SS Theoretical Physics

1 Exercises from the notes

(a)

5.1.1

$$\begin{aligned}
H(\{p_j\}) &\equiv -\sum_{j=1}^{N} p(x_j)\log_2[p(x_j)] \\
&= -\frac{1}{2}\log_2\left(\frac{1}{2}\right) - \frac{1}{4}\log_2\left(\frac{1}{4}\right) - \frac{1}{8}\log_2\left(\frac{1}{8}\right) - \frac{1}{8}\log_2\left(\frac{1}{8}\right) \\
&= -\frac{1}{2}\log_2\left(2^{-1}\right) - \frac{1}{4}\log_2\left(2^{-2}\right) - \frac{2}{8}\log_2\left(2^{-3}\right) \\
&= \frac{1}{2}\log_2(2) + \frac{2}{4}\log_2(2) + \frac{3}{4}\log_2(2) \\
&= \frac{7}{4}\log_2(2) \\
H(\{p_j\}) &= \frac{7}{4}
\end{aligned}$$

5.1.2

$$\begin{aligned}
H(X|Y) &= -\sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\log_2[p(x_j|y_k)] \\
&= -\sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\log_2\left[\frac{p(x_j,y_k)}{p(y_k)}\right] && \text{(as } p(x_j,y_k) = p(x_j|y_k)\,p(y_k)\text{)} \\
&= -\sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\left\{\log_2[p(x_j,y_k)] - \log_2[p(y_k)]\right\} \\
&= -\sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\log_2[p(x_j,y_k)] + \sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\log_2[p(y_k)] \\
&= H(X,Y) + \sum_{k=1}^{M}\left(\sum_{j=1}^{N} p(x_j,y_k)\right)\log_2[p(y_k)] \\
&= H(X,Y) - \left(-\sum_{k=1}^{M} p(y_k)\log_2[p(y_k)]\right) \\
H(X|Y) &= H(X,Y) - H(Y) && (1)
\end{aligned}$$

5.1.3

$$
\begin{aligned}
H(X,Y) &= -\sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\log_2[p(x_j,y_k)] \\
&= -\sum_{j=1}^{N}\sum_{k=1}^{M} \delta_{jk}p(y_k)\log_2[\delta_{jk}p(y_k)] \\
&= -\sum_{k=1}^{M} p(y_k)\log_2[p(y_k)] \\
H(X,Y) &= H(Y)
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
H(X,Y) &= -\sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\log_2[p(x_j,y_k)] \\
&= -\sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\log_2[p(y_k|x_j)\,p(x_j)] && (\text{as } p(x_j,y_k)=p(y_k|x_j)\,p(x_j)) \\
&= -\sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\left\{\log_2[p(y_k|x_j)]+\log_2[p(x_j)]\right\} \\
&= -\sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\log_2[p(y_k|x_j)] - \sum_{j=1}^{N}\sum_{k=1}^{M} p(x_j,y_k)\log_2[p(x_j)] \\
&= H(X|Y) - \sum_{j=1}^{N}\left(\sum_{k=1}^{M} p(x_j,y_k)\right)\log_2[p(x_j)] \\
&= H(X|Y) - \sum_{j=1}^{N} p(x_j)\log_2[p(x_j)]
\end{aligned}
$$

We have shown that $H(X|Y) = H(X,Y) - H(Y)$ (Equation 1). We have also shown that $H(X,Y) = H(Y)$ for perfectly correlated sources (Equation 2). Therefore for perfectly correlated sources, $H(X|Y) = H(Y) - H(Y) = 0$, and so we have

$$
\begin{aligned}
H(X,Y) &= 0 - \sum_{j=1}^{N} p(x_j)\log_2[p(x_j)] \\
&= H(X)
\end{aligned}
$$

Therefore $H(X,Y) = H(X) = H(Y)$.

5.1.4

$$
\begin{aligned}
H(X:Y) &= H(X) + H(Y) - H(X,Y) \\
&= H(X) + H(Y) - H(X|Y) - H(Y) && (\text{as } H(X|Y) = H(X,Y) - H(Y)) \\
&= H(X) - H(X|Y)
\end{aligned}
$$

Thus we can see that $H(X:Y)$ is maximised when $H(X|Y)$ is minimised, and minimised when $H(X|Y)$ is maximised.

$H(X|Y)$ obtains its minimum when there is no uncertainty in the value of X once Y is known, i.e. when X and Y are perfectly correlated. In this case $H(X|Y) = 0$, and so $H(X:Y) \leqslant H(X) - 0 = H(X)$.

$H(X|Y)$ obtains its maximum when there is no difference in the uncertainty of X when Y is known, i.e. when X and Y are independent. In this case $H(X|Y) = H(X)$, and so $H(X:Y) \geqslant H(X) - H(X) = 0$.

Therefore $0 \leqslant H(X:Y) \leqslant H(X)$, and $H(X:Y)$ is bounded below/above when X and Y are independent/perfectly correlated.

5.3.3

A	E	$p(E\|A)$	B	$p(E\|A)\,p(B\|EA)$
$0\ \hat{X}$	$0\ \hat{X}$	$\frac{1}{2}$	$0\ \hat{X}$	$\frac{1}{4}$
			$0\ \hat{Z}$	$\frac{1}{8}$
			$1\ \hat{Z}$	$\frac{1}{8}$
	$0\ \hat{Z}$	$\frac{1}{4}$	$0\ \hat{X}$	$\frac{1}{16}$
			$0\ \hat{Z}$	$\frac{1}{8}$
			$1\ \hat{X}$	$\frac{1}{16}$
	$1\ \hat{Z}$	$\frac{1}{4}$	$0\ \hat{X}$	$\frac{1}{16}$
			$1\ \hat{X}$	$\frac{1}{16}$
			$1\ \hat{Z}$	$\frac{1}{8}$
$0\ \hat{Z}$	$0\ \hat{X}$	$\frac{1}{4}$	$0\ \hat{X}$	$\frac{1}{8}$
			$0\ \hat{Z}$	$\frac{1}{16}$
			$1\ \hat{Z}$	$\frac{1}{16}$
	$0\ \hat{Z}$	$\frac{1}{2}$	$0\ \hat{X}$	$\frac{1}{8}$
			$0\ \hat{Z}$	$\frac{1}{4}$
			$1\ \hat{X}$	$\frac{1}{8}$
	$1\ \hat{X}$	$\frac{1}{4}$	$0\ \hat{Z}$	$\frac{1}{16}$
			$1\ \hat{X}$	$\frac{1}{8}$
			$1\ \hat{Z}$	$\frac{1}{16}$

A	E	$p(E\|A)$	B	$p(E\|A)\,p(B\|EA)$
$1\ \hat{X}$	$0\ \hat{Z}$	$\frac{1}{4}$	$0\ \hat{X}$	$\frac{1}{16}$
			$0\ \hat{Z}$	$\frac{1}{8}$
			$1\ \hat{X}$	$\frac{1}{16}$
	$1\ \hat{X}$	$\frac{1}{2}$	$0\ \hat{Z}$	$\frac{1}{8}$
			$1\ \hat{X}$	$\frac{1}{4}$
			$1\ \hat{Z}$	$\frac{1}{8}$
	$1\ \hat{Z}$	$\frac{1}{4}$	$0\ \hat{X}$	$\frac{1}{16}$
			$1\ \hat{X}$	$\frac{1}{16}$
			$1\ \hat{Z}$	$\frac{1}{8}$
$1\ \hat{Z}$	$0\ \hat{X}$	$\frac{1}{4}$	$0\ \hat{X}$	$\frac{1}{8}$
			$0\ \hat{Z}$	$\frac{1}{16}$
			$1\ \hat{Z}$	$\frac{1}{16}$
	$1\ \hat{X}$	$\frac{1}{4}$	$0\ \hat{Z}$	$\frac{1}{16}$
			$1\ \hat{X}$	$\frac{1}{8}$
			$1\ \hat{Z}$	$\frac{1}{16}$
	$1\ \hat{Z}$	$\frac{1}{2}$	$0\ \hat{X}$	$\frac{1}{8}$
			$1\ \hat{X}$	$\frac{1}{8}$
			$1\ \hat{Z}$	$\frac{1}{4}$

$$
\begin{aligned}
p(0=\mathcal{R}_i, 0=\mathcal{R}'_i) &= p(0=\mathcal{R}'_i|0=\mathcal{R}_i)\,p(0=\mathcal{R}_i)\\
&= \left(\frac{1}{4}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{4}+\frac{1}{16}\right)\frac{1}{2}\\
&= \frac{3}{8}\\
p(0=\mathcal{R}_i, 1=\mathcal{R}'_i) &= p(1=\mathcal{R}'_i|0=\mathcal{R}_i)\,p(0=\mathcal{R}_i)\\
&= \left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right)\frac{1}{2}\\
&= \frac{1}{8}\\
p(1=\mathcal{R}_i, 0=\mathcal{R}'_i) &= p(0=\mathcal{R}'_i|1=\mathcal{R}_i)\,p(1=\mathcal{R}_i)\\
&= \left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right)\frac{1}{2}\\
&= \frac{1}{8}\\
p(1=\mathcal{R}_i, 1=\mathcal{R}'_i) &= p(1=\mathcal{R}'_i|1=\mathcal{R}_i)\,p(1=\mathcal{R}_i)\\
&= \left(\frac{1}{16}+\frac{1}{4}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{4}\right)\frac{1}{2}\\
&= \frac{3}{8}
\end{aligned}
$$

$$
\begin{aligned}
H(X) = H(Y) &= -\frac{1}{2}\log_2\left(\frac{1}{2}\right) - \frac{1}{2}\log_2\left(\frac{1}{2}\right)\\
&= 1\\
H(X,Y) &= -\frac{3}{8}\log_2\left(\frac{3}{8}\right) - \frac{1}{8}\log_2\left(\frac{1}{8}\right) - \frac{1}{8}\log_2\left(\frac{1}{8}\right) - \frac{3}{8}\log_2\left(\frac{3}{8}\right)\\
&= 1.811\\
H(X:Y) &= H(X) + H(Y) - H(X,Y)\\
&= 0.189
\end{aligned}
$$

(b)

6.1.2

$$\begin{aligned}
\hat{\mathrm{H}}\hat{\mathrm{Y}}\hat{\mathrm{H}} &= \frac{1}{\sqrt{2}}\left(\hat{\mathrm{X}}+\hat{\mathrm{Z}}\right)\hat{\mathrm{Y}}\frac{1}{\sqrt{2}}\left(\hat{\mathrm{X}}+\hat{\mathrm{Z}}\right) \\
&= \frac{1}{2}\left(\hat{\mathrm{X}}\hat{\mathrm{Y}}\hat{\mathrm{X}}+\hat{\mathrm{X}}\hat{\mathrm{Y}}\hat{\mathrm{Z}}+\hat{\mathrm{Z}}\hat{\mathrm{Y}}\hat{\mathrm{X}}+\hat{\mathrm{Z}}\hat{\mathrm{Y}}\hat{\mathrm{X}}\right) \\
&= \frac{1}{2}\left(i\hat{\mathrm{Z}}\hat{\mathrm{X}}+i\hat{\mathrm{Z}}\hat{\mathrm{Z}}-i\hat{\mathrm{X}}\hat{\mathrm{X}}-i\hat{\mathrm{X}}\hat{\mathrm{Z}}\right) && \left(\text{as } \hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\mathbb{1}+i\textstyle\sum_{k=1}^{3}\varepsilon_{ijk}\hat{\sigma}_k\right) \\
&= \frac{1}{2}\left(-\hat{\mathrm{Y}}+i\mathbb{1}-i\mathbb{1}-\hat{\mathrm{Y}}\right) && \left(\text{as } \hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\mathbb{1}+i\textstyle\sum_{k=1}^{3}\varepsilon_{ijk}\hat{\sigma}_k\right) \\
\hat{\mathrm{H}}\hat{\mathrm{Y}}\hat{\mathrm{H}} &= -\hat{\mathrm{Y}}
\end{aligned}$$

$$\begin{aligned}
\hat{\mathrm{H}}\hat{\mathrm{Z}}\hat{\mathrm{H}} &= \frac{1}{\sqrt{2}}\left(\hat{\mathrm{X}}+\hat{\mathrm{Z}}\right)\hat{\mathrm{Z}}\frac{1}{\sqrt{2}}\left(\hat{\mathrm{X}}+\hat{\mathrm{Z}}\right) \\
&= \frac{1}{2}\left(\hat{\mathrm{X}}\hat{\mathrm{Z}}\hat{\mathrm{X}}+\hat{\mathrm{X}}\hat{\mathrm{Z}}\hat{\mathrm{Z}}+\hat{\mathrm{Z}}\hat{\mathrm{Z}}\hat{\mathrm{X}}+\hat{\mathrm{Z}}\hat{\mathrm{Z}}\hat{\mathrm{Z}}\right) \\
&= \frac{1}{2}\left(i\hat{\mathrm{X}}\hat{\mathrm{Y}}+\hat{\mathrm{X}}+\hat{\mathrm{X}}+\hat{\mathrm{Z}}\right) && \left(\text{as } \hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\mathbb{1}+i\textstyle\sum_{k=1}^{3}\varepsilon_{ijk}\hat{\sigma}_k\right) \\
&= \frac{1}{2}\left(-\hat{\mathrm{Z}}+2\hat{\mathrm{X}}+\hat{\mathrm{Z}}\right) && \left(\text{as } \hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\mathbb{1}+i\textstyle\sum_{k=1}^{3}\varepsilon_{ijk}\hat{\sigma}_k\right) \\
&= \hat{\mathrm{X}}
\end{aligned}$$

$$\begin{aligned}
\hat{\mathrm{H}}\hat{\mathrm{Z}}\hat{\mathrm{H}} = \hat{\mathrm{X}} &\implies \hat{\mathrm{H}}\hat{\mathrm{H}}\hat{\mathrm{Z}}\hat{\mathrm{H}}\hat{\mathrm{H}} = \hat{\mathrm{H}}\hat{\mathrm{X}}\hat{\mathrm{H}} \\
&\implies \hat{\mathrm{Z}} = \hat{\mathrm{H}}\hat{\mathrm{X}}\hat{\mathrm{H}} && (\text{as } \hat{\mathrm{H}}\hat{\mathrm{H}} = \mathbb{1})
\end{aligned}$$

As we have that

$$\begin{aligned}
\hat{\mathrm{X}}\left|0\right\rangle &= \left|1\right\rangle & \hat{\mathrm{Z}}\left|+\right\rangle &= \left|-\right\rangle \\
\hat{\mathrm{X}}\left|1\right\rangle &= \left|0\right\rangle & \hat{\mathrm{Z}}\left|-\right\rangle &= \left|+\right\rangle,
\end{aligned}$$

the Pauli gates $\hat{\mathrm{X}}$ and $\hat{\mathrm{Z}}$ can be considered as a bit-flip on the basis states of $\hat{\mathrm{Z}}$ and $\hat{\mathrm{X}}$, respectively. Therefore both $\hat{\mathrm{H}}\hat{\mathrm{Z}}\hat{\mathrm{H}} = \hat{\mathrm{X}}$ and $\hat{\mathrm{H}}\hat{\mathrm{X}}\hat{\mathrm{H}} = \hat{\mathrm{Z}}$ state that a bit-flip on the basis states of one of the Pauli gates $\hat{\mathrm{X}}$, $\hat{\mathrm{Z}}$ can be related to a bit-flip on the basis states of the other gate via left and right multiplication of Hadamard gates $\hat{\mathrm{H}}$.

6.2.1

$$\begin{aligned}
\mathbb{1}\otimes\hat{\mathrm{H}} &= \begin{pmatrix} \hat{\mathrm{H}} & \mathbb{0} \\ \mathbb{0} & \hat{\mathrm{H}} \end{pmatrix} & \hat{\mathrm{H}}\otimes\mathbb{1} &= \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} & &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}
\end{aligned}$$

6.2.3

$$
\begin{aligned}
\left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) (|x\rangle \otimes |y\rangle) &= \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \left(\hat{\mathrm{H}} |x\rangle \otimes \hat{\mathrm{H}} |y\rangle\right) \\
&= \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \frac{1}{\sqrt{2}} (|0\rangle + (-1)^x |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle + (-1)^y |1\rangle) \\
&\qquad\qquad \left(\text{as } \hat{\mathrm{H}} |x\rangle = \tfrac{1}{\sqrt{2}} (|0\rangle + (-1)^x |1\rangle)\right) \\
&= \frac{1}{2} \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \big(|0\rangle \otimes |0\rangle + (-1)^y |0\rangle \otimes |1\rangle \\
&\qquad + (-1)^x |1\rangle \otimes |0\rangle + (-1)^{x+y} |1\rangle \otimes |1\rangle\big) \\
&= \frac{1}{2} \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \big(|0\rangle \otimes |0\rangle + (-1)^y |0\rangle \otimes |1\rangle \\
&\qquad + (-1)^x |1\rangle \otimes |1\rangle + (-1)^{x+y} |1\rangle \otimes |0\rangle\big) \\
&\qquad\qquad \left(\text{as } \hat{\mathrm{C}}_X |x\rangle \otimes |y\rangle = |x\rangle \otimes |x \oplus y\rangle\right) \\
&= \frac{1}{2} \Big(\hat{\mathrm{H}} |0\rangle \otimes \hat{\mathrm{H}} |0\rangle + (-1)^y \hat{\mathrm{H}} |0\rangle \otimes \hat{\mathrm{H}} |1\rangle \\
&\qquad + (-1)^{x+y} \hat{\mathrm{H}} |1\rangle \otimes \hat{\mathrm{H}} |0\rangle + (-1)^x \hat{\mathrm{H}} |1\rangle \otimes \hat{\mathrm{H}} |1\rangle\Big) \\
&= \frac{1}{2} \left[\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)\right. \\
&\qquad + (-1)^y \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \\
&\qquad + (-1)^{x+y} \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\
&\qquad \left. + (-1)^x \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)\right] \\
&\qquad\qquad \left(\text{as } \hat{\mathrm{H}} |x\rangle = \tfrac{1}{\sqrt{2}} (|0\rangle + (-1)^x |1\rangle)\right) \\
&= \frac{1}{4} \Big[\left(1 + (-1)^x + (-1)^y + (-1)^{x+y}\right) |0\rangle \otimes |0\rangle \\
&\qquad + \left(1 - (-1)^x - (-1)^y + (-1)^{x+y}\right) |0\rangle \otimes |1\rangle \\
&\qquad + \left(1 - (-1)^x + (-1)^y - (-1)^{x+y}\right) |1\rangle \otimes |0\rangle \\
&\qquad + \left(1 + (-1)^x - (-1)^y - (-1)^{x+y}\right) |1\rangle \otimes |1\rangle\Big] \\
\left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) (|0\rangle \otimes |0\rangle) &= |0\rangle \otimes |0\rangle \\
\left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) (|0\rangle \otimes |1\rangle) &= |1\rangle \otimes |1\rangle \\
\left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) (|1\rangle \otimes |0\rangle) &= |1\rangle \otimes |0\rangle \\
\left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) (|1\rangle \otimes |1\rangle) &= |0\rangle \otimes |1\rangle \\
\implies \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) (|x\rangle \otimes |y\rangle) &= \delta_0^x \delta_0^y |0\rangle \otimes |0\rangle + \delta_1^x \delta_1^y |0\rangle \otimes |1\rangle + \delta_1^x \delta_0^y |1\rangle \otimes |0\rangle + \delta_0^x \delta_1^y |1\rangle \otimes |1\rangle
\end{aligned}
$$

$$
\begin{aligned}
\left(\mathbb{1} \otimes |0\rangle \langle 0| + \hat{\mathrm{X}} \otimes |1\rangle \langle 1|\right) (|x\rangle \otimes |y\rangle) &= |x\rangle \otimes |0\rangle \langle 0 \,|\, y\rangle + \hat{\mathrm{X}} |x\rangle \otimes |1\rangle \langle 1 \,|\, y\rangle \\
&= \langle 0 \,|\, y\rangle |x\rangle \otimes |0\rangle + \langle 1 \,|\, y\rangle |x + 1\rangle \otimes |1\rangle \\
&= \delta_0^y \left(\delta_0^x |0\rangle \otimes |0\rangle + \delta_1^x |1\rangle \otimes |0\rangle\right) + \delta_1^y \left(\delta_0^x |1\rangle \otimes |1\rangle + \delta_1^x |0\rangle \otimes |1\rangle\right) \\
&= \delta_0^x \delta_0^y |0\rangle \otimes |0\rangle + \delta_1^x \delta_1^y |0\rangle \otimes |1\rangle + \delta_1^x \delta_0^y |1\rangle \otimes |0\rangle + \delta_0^x \delta_1^y |1\rangle \otimes |1\rangle \\
&= \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) (|x\rangle \otimes |y\rangle) \\
\implies \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) \hat{\mathrm{C}}_X \left(\hat{\mathrm{H}} \otimes \hat{\mathrm{H}}\right) &= \left(\mathbb{1} \otimes |0\rangle \langle 0| + \hat{\mathrm{X}} \otimes |1\rangle \langle 1|\right)
\end{aligned}
$$

Thus the circuit relation in Figure 6.5(b) holds.

6.2.4

$$
\begin{aligned}
\left(\hat{\mathrm{H}}\otimes\hat{\mathrm{H}}\right)\hat{\mathrm{C}}_X\left(\hat{\mathrm{H}}\otimes\hat{\mathrm{H}}\right) &= \left(\hat{\mathrm{H}}\otimes\hat{\mathrm{H}}\right)\left(|0\rangle\langle 0|\otimes\mathbb{1}+|1\rangle\langle 1|\otimes\hat{\mathrm{X}}\right)\left(\hat{\mathrm{H}}\otimes\hat{\mathrm{H}}\right)\\
&= \hat{\mathrm{H}}|0\rangle\langle 0|\hat{\mathrm{H}}\otimes\hat{\mathrm{H}}\mathbb{1}\hat{\mathrm{H}}+\hat{\mathrm{H}}|1\rangle\langle 1|\hat{\mathrm{H}}\otimes\hat{\mathrm{H}}\hat{\mathrm{X}}\hat{\mathrm{H}}\\
&= \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\frac{1}{\sqrt{2}}(\langle 0|+\langle 1|)\otimes\mathbb{1}+\frac{1}{\sqrt{2}}(\langle 0|-\langle 1|)\frac{1}{\sqrt{2}}(\langle 0|-\langle 1|)\otimes\hat{\mathrm{Z}}\\
&\qquad \left(\text{as } \hat{\mathrm{H}}\hat{\mathrm{H}}=\mathbb{1},\ \hat{\mathrm{H}}\hat{\mathrm{X}}\hat{\mathrm{H}}=\hat{\mathrm{Z}},\ \hat{\mathrm{H}}|0\rangle=\tfrac{1}{\sqrt{2}}(|0\rangle+|1\rangle),\ \hat{\mathrm{H}}|1\rangle=\tfrac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right)\\
&= \frac{1}{2}(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|+|1\rangle\langle 1|)\otimes(|0\rangle\langle 0|+|1\rangle\langle 1|)\\
&\quad +\frac{1}{2}(|0\rangle\langle 0|-|0\rangle\langle 1|-|1\rangle\langle 0|+|1\rangle\langle 1|)\otimes(|0\rangle\langle 0|-|1\rangle\langle 1|)\\
&= \frac{1}{2}(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|+|1\rangle\langle 1|+|0\rangle\langle 0|-|0\rangle\langle 1|-|1\rangle\langle 0|+|1\rangle\langle 1|)\otimes|0\rangle\langle 0|\\
&\quad +\frac{1}{2}(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|+|1\rangle\langle 1|-|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|-|1\rangle\langle 1|)\otimes|1\rangle\langle 1|\\
&= (|0\rangle\langle 0|+|1\rangle\langle 1|)\otimes|0\rangle\langle 0|+(|0\rangle\langle 1|+|1\rangle\langle 0|)\otimes|1\rangle\langle 1|\\
&= \mathbb{1}\otimes|0\rangle\langle 0|+\hat{\mathrm{X}}\otimes|1\rangle\langle 1|
\end{aligned}
$$

Thus the circuit relation in Figure 6.5(b) holds.

$$
\begin{aligned}
\hat{\mathrm{I}}_\phi &= \hat{\mathrm{C}}_{e^{i\phi}\mathbb{1}}\\
&= |0\rangle\langle 0|\otimes\mathbb{1}+|1\rangle\langle 1|\otimes e^{i\phi}\mathbb{1} && \left(\text{as } \hat{\mathrm{C}}_U\equiv|0\rangle\langle 0|\otimes\mathbb{1}+|1\rangle\langle 1|\otimes U\right)\\
&= \left(|0\rangle\langle 0|+e^{i\phi}|1\rangle\langle 1|\right)\otimes\mathbb{1}\\
&= \hat{\mathrm{P}}_\phi\otimes\mathbb{1}
\end{aligned}
$$

Thus the circuit relation in Figure 6.5(d) holds.

(c)

7.1.7

The inner product for $\hat{\mathrm{A}}$ and $\hat{\mathrm{B}}$ in the space generated by the Pauli basis $\left\{\mathbb{1},\hat{\mathrm{X}},\hat{\mathrm{Y}},\hat{\mathrm{Z}}\right\}$ is defined as

$$\mathcal{S}\left(\hat{\mathrm{A}},\hat{\mathrm{B}}\right)=\mathrm{tr}\left(\hat{\mathrm{A}}^\dagger\hat{\mathrm{B}}\right).$$

First consider $\hat{\mathrm{A}}=\mathbb{1}$, $\hat{\mathrm{B}}\in\left\{\hat{\mathrm{X}},\hat{\mathrm{Y}},\hat{\mathrm{Z}}\right\}$. Then

$$
\begin{aligned}
\mathcal{S}\left(\mathbb{1},\hat{\mathrm{B}}\right) &= \mathrm{tr}\left(\mathbb{1}^\dagger\hat{\mathrm{B}}\right)\\
&= \mathrm{tr}\left(\hat{\mathrm{B}}\right)\\
&= 0. && (\text{as the Pauli matrices are traceless})
\end{aligned}
$$

As $\mathcal{S}\left(\hat{\mathrm{B}},\hat{\mathrm{A}}\right)=\mathcal{S}\left(\hat{\mathrm{A}},\hat{\mathrm{B}}\right)^*$ then we also have $\mathcal{S}\left(\hat{\mathrm{B}},\mathbb{1}\right)=0$, and so $\mathbb{1}$ is orthogonal to the Pauli matrices.

Now consider $\hat{\mathrm{A}},\hat{\mathrm{B}}\in\left\{\hat{\mathrm{X}},\hat{\mathrm{Y}},\hat{\mathrm{Z}}\right\}$, $\hat{\mathrm{A}}\neq\hat{\mathrm{B}}$, and denote $\hat{\mathrm{C}}\in\left\{\hat{\mathrm{X}},\hat{\mathrm{Y}},\hat{\mathrm{Z}}\right\}$ where $\hat{\mathrm{A}}\neq\hat{\mathrm{C}}\neq\hat{\mathrm{B}}$. Then

$$
\begin{aligned}
\mathcal{S}\left(\hat{\mathrm{A}},\hat{\mathrm{B}}\right) &= \mathrm{tr}\left(\hat{\mathrm{A}}^\dagger\hat{\mathrm{B}}\right)\\
&= \mathrm{tr}\left(\hat{\mathrm{A}}\hat{\mathrm{B}}\right) && (\text{as the Pauli matrices are Hermitian})\\
&= \mathrm{tr}\left(\pm i\hat{\mathrm{C}}\right) && \left(\text{as } \hat{\sigma}_i\hat{\sigma}_j=\delta_{ij}\mathbb{1}+i\textstyle\sum_{k=1}^3\varepsilon_{ijk}\hat{\sigma}_k\right)\\
&= \pm i\,\mathrm{tr}\left(\hat{\mathrm{C}}\right)\\
&= 0. && (\text{as the Pauli matrices are traceless})
\end{aligned}
$$

Thus the Pauli matrices are orthogonal to each other.

Therefore each element of the Pauli basis is orthogonal to the others, and so the basis is orthogonal.

7.1.8

$$\begin{aligned}
\hat{\rho}^2 &= \frac{1}{2}\left(\mathbb{1} + n_x\hat{\mathrm{X}} + n_y\hat{\mathrm{Y}} + n_z\hat{\mathrm{Z}}\right)\frac{1}{2}\left(\mathbb{1} + n_x\hat{\mathrm{X}} + n_y\hat{\mathrm{Y}} + n_z\hat{\mathrm{Z}}\right) \\
&= \frac{1}{4}\Big(\mathbb{1} + n_x\hat{\mathrm{X}} + n_y\hat{\mathrm{Y}} + n_z\hat{\mathrm{Z}} + n_x\hat{\mathrm{X}} + n_x^2\hat{\mathrm{X}}^2 + n_x n_y\hat{\mathrm{X}}\hat{\mathrm{Y}} + n_x n_z\hat{\mathrm{X}}\hat{\mathrm{Z}} \\
&\qquad + n_y\hat{\mathrm{Y}} + n_y n_x\hat{\mathrm{Y}}\hat{\mathrm{X}} + n_y^2\hat{\mathrm{Y}}^2 + n_y n_z\hat{\mathrm{Y}}\hat{\mathrm{Z}} + n_z\hat{\mathrm{Z}} + n_z n_x\hat{\mathrm{Z}}\hat{\mathrm{X}} + n_z n_y\hat{\mathrm{Z}}\hat{\mathrm{Y}} + n_z^2\hat{\mathrm{Z}}^2\Big) \\
&= \frac{1}{4}\left[\left(1 + n_x^2 + n_y^2 + n_z^2\right)\mathbb{1} + n_x\hat{\mathrm{X}} + n_y\hat{\mathrm{Y}} + n_z\hat{\mathrm{Z}} + n_x n_y\left\{\hat{\mathrm{X}}, \hat{\mathrm{Y}}\right\} + n_y n_z\left\{\hat{\mathrm{Y}}, \hat{\mathrm{Z}}\right\} + n_z n_x\left\{\hat{\mathrm{Z}}, \hat{\mathrm{X}}\right\}\right] \\
&&& \text{(as } \hat{\sigma}_i^2 = \mathbb{1}\text{)} \\
&= \frac{1}{4}\left[\left(1 + ||\mathbf{n}||^2\right)\mathbb{1} + n_x\hat{\mathrm{X}} + n_y\hat{\mathrm{Y}} + n_z\hat{\mathrm{Z}}\right] && \text{(as } \{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{i,j}\mathbb{1}\text{)} \\
\operatorname{tr}\left(\hat{\rho}^2\right) &= \frac{1}{4}\operatorname{tr}\left[\left(1 + ||\mathbf{n}||^2\right)\mathbb{1} + n_x\hat{\mathrm{X}} + n_y\hat{\mathrm{Y}} + n_z\hat{\mathrm{Z}}\right] \\
&= \frac{1}{4}\left[\left(1 + ||\mathbf{n}||^2\right)\operatorname{tr}(\mathbb{1}) + n_x\operatorname{tr}\left(\hat{\mathrm{X}}\right) + n_y\operatorname{tr}\left(\hat{\mathrm{Y}}\right) + n_z\operatorname{tr}\left(\hat{\mathrm{Z}}\right)\right] \\
&= \frac{1 + ||\mathbf{n}||^2}{2} && \text{(as } \operatorname{tr}(\mathbb{1}) = 2,\ \operatorname{tr}(\hat{\sigma}_i) = 0\text{)}
\end{aligned}$$

For $\hat{\rho}$ to be a density operator, we require that $0 \leqslant ||\mathbf{n}|| \leqslant 1$. If $||\mathbf{n}|| = 0$ then we have that $\hat{\rho} = \frac{1}{2}\mathbb{1}$ corresponds to a maximally mixed state, and $\operatorname{tr}\left(\hat{\rho}^2\right) = \frac{1}{2}$. If $||\mathbf{n}|| = 1$ then we have that $\hat{\rho} = \frac{1}{2}\left(\mathbb{1} + \hat{\mathrm{X}} + \hat{\mathrm{Y}} + \hat{\mathrm{Z}}\right)$ corresponds to a pure state, and $\operatorname{tr}\left(\hat{\rho}^2\right) = 1$. Therefore we have that $\operatorname{tr}\left(\hat{\rho}^2\right) = 1$ for pure states, and $\frac{1}{2} \leqslant \operatorname{tr}\left(\hat{\rho}^2\right) < 1$ for mixed states.

(d)

8.2.1

$$\begin{aligned}
\mathbb{B}_{\mathrm{qm}}(\theta) &= -\mathbf{a}\cdot\mathbf{b} - \mathbf{a}'\cdot\mathbf{b}' - \mathbf{a}'\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b}' \\
&= -\cos\pi - \cos 2\theta - \cos(\pi - \theta) + \cos\theta && \text{(from Figure 8.5(a))} \\
&= 1 - \cos 2\theta - \cos\pi\cos\theta - \sin\pi\sin\theta + \cos\theta \\
&= 1 - \cos 2\theta + \cos\theta - 0 + \cos\theta \\
&= 1 + 2\cos\theta - \cos 2\theta
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{d\mathbb{B}_{\mathrm{qm}}(\theta)}{d\theta} \\
&= -2\sin\theta + 2\sin 2\theta \\
\implies \sin\theta &= \sin 2\theta \\
\implies \sin\theta &= 2\sin\theta\cos\theta
\end{aligned}$$

$$\begin{aligned}
&\sin 0 = 2\sin 0\cos 0 = 0 && 1 = 2\cos\theta \text{ for } \theta \neq 0 \\
\implies &\text{min./max. at } \theta = 0 && \implies \text{min./max. at } \cos\theta = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\mathbb{B}_{\mathrm{qm}}(0) &= 1 + 2 - 1 \\
&= 2 \ngtr 2 \implies \text{not violated for } \theta = 0 \\
\mathbb{B}_{\mathrm{qm}}\left(\arccos\frac{1}{2}\right) &= 1 + 1 + \frac{1}{2} \\
&= \frac{5}{2} > 2 \implies \text{maximum violation of } \frac{5}{2} \text{ at } \cos\theta = \frac{1}{2}
\end{aligned}$$

8.3.1

$$
\begin{aligned}
\mathbb{E}(\mathbf{a}'', \mathbf{b}'') &= \frac{1}{\sqrt{2}} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \left(-\frac{\hat{\mathrm{X}} + \hat{\mathrm{Z}}}{\sqrt{2}} \otimes \hat{\mathrm{Z}}\right) \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right) \\
&= -\frac{1}{2\sqrt{2}} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \Big(\hat{\mathrm{X}}\,|0\rangle \otimes \hat{\mathrm{Z}}\,|1\rangle - \hat{\mathrm{X}}\,|1\rangle \otimes \hat{\mathrm{Z}}\,|0\rangle + \hat{\mathrm{Z}}\,|0\rangle \otimes \hat{\mathrm{Z}}\,|1\rangle - \hat{\mathrm{Z}}\,|1\rangle \otimes \hat{\mathrm{Z}}\,|0\rangle\Big) \\
&= -\frac{1}{2\sqrt{2}} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \left(-|1\rangle \otimes |1\rangle - |0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\right) \\
&= -\frac{1}{2\sqrt{2}} \left(-0 - 0 - 1 - 1\right) \\
&= \frac{1}{\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(\mathbf{a}', \mathbf{b}') &= \frac{1}{\sqrt{2}} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \left(\hat{\mathrm{X}} \otimes \frac{\hat{\mathrm{Z}} - \hat{\mathrm{X}}}{\sqrt{2}}\right) \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right) \\
&= \frac{1}{2\sqrt{2}} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \Big(\hat{\mathrm{X}}\,|0\rangle \otimes \hat{\mathrm{Z}}\,|1\rangle - \hat{\mathrm{X}}\,|1\rangle \otimes \hat{\mathrm{Z}}\,|0\rangle - \hat{\mathrm{X}}\,|0\rangle \otimes \hat{\mathrm{X}}\,|1\rangle + \hat{\mathrm{X}}\,|1\rangle \otimes \hat{\mathrm{X}}\,|0\rangle\Big) \\
&= \frac{1}{2\sqrt{2}} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \left(-|1\rangle \otimes |1\rangle - |0\rangle \otimes |0\rangle - |1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle\right) \\
&= \frac{}{2\sqrt{2}} \left(-0 - 0 + 1 + 1\right) \\
&= \frac{1}{\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(\mathbf{a}', \mathbf{b}'') &= \frac{1}{\sqrt{2}} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \Big(\hat{\mathrm{X}} \otimes \hat{\mathrm{Z}}\Big) \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right) \\
&= \frac{1}{2} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \Big(\hat{\mathrm{X}}\,|0\rangle \otimes \hat{\mathrm{Z}}\,|1\rangle - \hat{\mathrm{X}}\,|1\rangle \otimes \hat{\mathrm{Z}}\,|0\rangle\Big) \\
&= \frac{1}{2} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \left(-|1\rangle \otimes |1\rangle - |0\rangle \otimes |0\rangle\right) \\
&= \frac{1}{2} \left(-0 - 0\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(\mathbf{a}'', \mathbf{b}') &= \frac{1}{\sqrt{2}} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \left(-\frac{\hat{\mathrm{X}} + \hat{\mathrm{Z}}}{\sqrt{2}} \otimes \frac{\hat{\mathrm{X}} - \hat{\mathrm{Z}}}{\sqrt{2}}\right) \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right) \\
&= -\frac{1}{4} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \Big(\hat{\mathrm{X}}\,|0\rangle \otimes \hat{\mathrm{X}}\,|1\rangle - \hat{\mathrm{X}}\,|1\rangle \otimes \hat{\mathrm{X}}\,|0\rangle - \hat{\mathrm{X}}\,|0\rangle \otimes \hat{\mathrm{Z}}\,|1\rangle + \hat{\mathrm{X}}\,|1\rangle \otimes \hat{\mathrm{Z}}\,|0\rangle \\
&\qquad + \hat{\mathrm{Z}}\,|0\rangle \otimes \hat{\mathrm{X}}\,|1\rangle - \hat{\mathrm{Z}}\,|1\rangle \otimes \hat{\mathrm{X}}\,|0\rangle - \hat{\mathrm{Z}}\,|0\rangle \otimes \hat{\mathrm{Z}}\,|1\rangle + \hat{\mathrm{Z}}\,|1\rangle \otimes \hat{\mathrm{Z}}\,|0\rangle\Big) \\
&= -\frac{1}{4} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \big(|1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle \\
&\qquad + |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\big) \\
&= -\frac{1}{4} \left(\langle 0| \otimes \langle 1| - \langle 1| \otimes \langle 0|\right) \left(2\,|0\rangle \otimes |0\rangle + 2\,|1\rangle \otimes |1\rangle\right) \\
&= -\frac{1}{2} \left(0 + 0\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{B}_{\mathrm{qm}} &= \mathbb{E}(\mathbf{a}'', \mathbf{b}'') + \mathbb{E}(\mathbf{a}', \mathbf{b}') + \mathbb{E}(\mathbf{a}', \mathbf{b}'') - \mathbb{E}(\mathbf{a}'', \mathbf{b}') \\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 - 0 \\
&= \sqrt{2}
\end{aligned}
$$

2 Approximate Quantum Cloning

(a)

$$
\begin{aligned}
& && |0\rangle \otimes |0\rangle \\
&\text{Apply } R_y(-2\theta_1) \otimes \mathbb{1} && \to (\cos\theta_1 |0\rangle - \sin\theta_1 |1\rangle) \otimes |0\rangle \\
&\text{Apply } |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \hat{\mathrm{X}} && \to \cos\theta_1 |0\rangle \otimes |0\rangle - \sin\theta_1 |1\rangle \otimes |1\rangle \\
&\text{Apply } \mathbb{1} \otimes R_y(-2\theta_2) && \to \cos\theta_1 |0\rangle \otimes (\cos\theta_2 |0\rangle - \sin\theta_2 |1\rangle) \\
& && \quad - \sin\theta_1 |1\rangle \otimes (\sin\theta_2 |0\rangle + \cos\theta_2 |1\rangle) \\
&\text{Apply } \mathbb{1} \otimes |0\rangle\langle 0| + \hat{\mathrm{X}} \otimes |1\rangle\langle 1| && \to \cos\theta_1 \cos\theta_2 |0\rangle \otimes |0\rangle - \sin\theta_1 \sin\theta_2 |1\rangle \otimes |0\rangle \\
& && \quad - \cos\theta_1 \sin\theta_2 |1\rangle \otimes |1\rangle - \sin\theta_1 \cos\theta_2 |0\rangle \otimes |1\rangle \\
& && = \cos\theta_2 |0\rangle \otimes (\cos\theta_1 |0\rangle - \sin\theta_1 |1\rangle) \\
& && \quad - \sin\theta_2 |1\rangle \otimes (\sin\theta_1 |0\rangle + \cos\theta_1 |1\rangle) \\
&\text{Apply } R_y(-2\theta_3) \otimes \mathbb{1} && \to \cos\theta_2 (\cos\theta_3 |0\rangle - \sin\theta_3 |1\rangle) \otimes (\cos\theta_1 |0\rangle - \sin\theta_1 |1\rangle) \\
& && \quad - \sin\theta_2 (\sin\theta_3 |0\rangle + \cos\theta_3 |1\rangle) \otimes (\sin\theta_1 |0\rangle + \cos\theta_1 |1\rangle) \\
& && = (\cos\theta_1 \cos\theta_2 \cos\theta_3 - \sin\theta_1 \sin\theta_2 \sin\theta_3) |0\rangle \otimes |0\rangle \\
& && \quad - (\sin\theta_1 \cos\theta_2 \cos\theta_3 + \cos\theta_1 \sin\theta_2 \sin\theta_3) |0\rangle \otimes |1\rangle \\
& && \quad - (\cos\theta_1 \cos\theta_2 \sin\theta_3 + \sin\theta_1 \sin\theta_2 \cos\theta_3) |1\rangle \otimes |0\rangle \\
& && \quad + (\sin\theta_1 \cos\theta_2 \sin\theta_3 - \cos\theta_1 \sin\theta_2 \cos\theta_3) |1\rangle \otimes |1\rangle \\
& && = \sqrt{\frac{2}{3}} |0\rangle \otimes |0\rangle + \frac{1}{\sqrt{6}} |0\rangle \otimes |1\rangle + 0 |1\rangle \otimes |0\rangle + \frac{1}{\sqrt{6}} |1\rangle \otimes |1\rangle \\
& && = \frac{1}{\sqrt{6}} (2 |00\rangle + |01\rangle + |11\rangle)
\end{aligned}
$$

(b)

$$
\begin{aligned}
& && |\Psi\rangle \otimes |\Phi\rangle \\
& && = \frac{1}{\sqrt{6}} \left[(\alpha |0\rangle + \beta |1\rangle) \otimes (2|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle) \right] \\
&\text{Apply } \left(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \hat{\mathrm{X}} \right) \otimes \mathbb{1} && \to \frac{\alpha}{\sqrt{6}} |0\rangle \otimes (2|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle) \\
& && \quad + \frac{\beta}{\sqrt{6}} |1\rangle \otimes (2|1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle) \\
&\text{Apply } |0\rangle\langle 0| \otimes \mathbb{1} \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathbb{1} \otimes \hat{\mathrm{X}} && \to \frac{\alpha}{\sqrt{6}} |0\rangle \otimes (2|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle) \\
& && \quad + \frac{\beta}{\sqrt{6}} |1\rangle \otimes (2|1\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |0\rangle \otimes |0\rangle) \\
&\text{Apply } \left(\mathbb{1} \otimes |0\rangle\langle 0| + \hat{\mathrm{X}} \otimes |1\rangle\langle 1| \right) \otimes \mathbb{1} && \to \frac{\alpha}{\sqrt{6}} |0\rangle \otimes (2|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle) + \frac{\beta}{\sqrt{6}} |1\rangle \otimes |0\rangle \otimes |0\rangle \\
& && \quad + \frac{\alpha}{\sqrt{6}} |1\rangle \otimes |1\rangle \otimes |1\rangle + \frac{\beta}{\sqrt{6}} |0\rangle \otimes (2|1\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \\
& && = \frac{1}{\sqrt{6}} |0\rangle \otimes \left[\alpha (2|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle) + \beta (2|1\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \right] \\
& && \quad + \frac{1}{\sqrt{6}} |1\rangle \otimes (\alpha |1\rangle \otimes |1\rangle + \beta |0\rangle \otimes |0\rangle) \\
&\text{Apply } \mathbb{1} \otimes \mathbb{1} \otimes |0\rangle\langle 0| + \hat{\mathrm{X}} \otimes \mathbb{1} \otimes |1\rangle\langle 1| && \to \frac{1}{\sqrt{6}} \left[|0\rangle \otimes (2\alpha |0\rangle \otimes |0\rangle + \beta |1\rangle \otimes |0\rangle) + \beta |1\rangle \otimes |0\rangle \otimes |0\rangle \right] \\
& && \quad + \frac{1}{\sqrt{6}} \left[|1\rangle \otimes (\alpha |0\rangle \otimes |1\rangle + 2\beta |1\rangle \otimes |1\rangle) + \alpha |0\rangle \otimes |1\rangle \otimes |1\rangle \right] \\
& && = \frac{1}{\sqrt{6}} (2\alpha |0\rangle \otimes |0\rangle + \beta |0\rangle \otimes |1\rangle + \beta |1\rangle \otimes |0\rangle) \otimes |0\rangle \\
& && \quad + \frac{1}{\sqrt{6}} (\alpha |1\rangle \otimes |0\rangle + 2\beta |1\rangle \otimes |1\rangle + \alpha |0\rangle \otimes |1\rangle) \otimes |1\rangle \\
& && = \left[\alpha \sqrt{\frac{2}{3}} |00\rangle + \frac{\beta}{\sqrt{6}} (|10\rangle + |01\rangle) \right] |0\rangle \\
& && \quad + \left[\beta \sqrt{\frac{2}{3}} |11\rangle + \frac{\alpha}{\sqrt{6}} (|10\rangle + |01\rangle) \right] |1\rangle
\end{aligned}
$$

(c)

$$
\begin{aligned}
\hat{\rho} =\ & |\Psi\Phi\rangle \langle \Psi\Phi| \\
=\ & \left\{ \left[\alpha \sqrt{\frac{2}{3}} |00\rangle + \frac{\beta}{\sqrt{6}} (|10\rangle + |01\rangle) \right] |0\rangle + \left[\beta \sqrt{\frac{2}{3}} |11\rangle + \frac{\alpha}{\sqrt{6}} (|10\rangle + |01\rangle) \right] |1\rangle \right\} \times \\
& \times \left\{ \langle 0| \left[\alpha \sqrt{\frac{2}{3}} \langle 00| + \frac{\beta}{\sqrt{6}} (\langle 10| + \langle 01|) \right] + \langle 1| \left[\beta \sqrt{\frac{2}{3}} \langle 11| + \frac{\alpha}{\sqrt{6}} (\langle 10| + \langle 01|) \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}_{12} =\ & \mathrm{tr}_3(\hat{\rho}) \\
=\ & \left(\alpha \sqrt{\frac{2}{3}} |00\rangle + \frac{\beta}{\sqrt{6}} |10\rangle + \frac{\beta}{\sqrt{6}} |01\rangle \right) \left(\alpha \sqrt{\frac{2}{3}} \langle 00| + \frac{\beta}{\sqrt{6}} \langle 10| + \frac{\beta}{\sqrt{6}} \langle 01| \right) + 0 \\
& + 0 + \left(\beta \sqrt{\frac{2}{3}} |11\rangle + \frac{\alpha}{\sqrt{6}} |10\rangle + \frac{\alpha}{\sqrt{6}} |01\rangle \right) \left(\beta \sqrt{\frac{2}{3}} \langle 11| + \frac{\alpha}{\sqrt{6}} \langle 10| + \frac{\alpha}{\sqrt{6}} \langle 01| \right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}_1 &= \operatorname{tr}_2(\hat{\rho}_{12}) \\
&= \left(\alpha\sqrt{\frac{2}{3}}\left|0\right\rangle + \frac{\beta}{\sqrt{6}}\left|1\right\rangle\right)\left(\alpha\sqrt{\frac{2}{3}}\left\langle 0\right| + \frac{\beta}{\sqrt{6}}\left\langle 1\right|\right) + \frac{\beta}{\sqrt{6}}\left|0\right\rangle\frac{\beta}{\sqrt{6}}\left\langle 0\right| + 0 \\
&\quad + \frac{\alpha}{\sqrt{6}}\left|1\right\rangle\frac{\alpha}{\sqrt{6}}\left\langle 1\right| + \left(\beta\sqrt{\frac{2}{3}}\left|1\right\rangle + \frac{\alpha}{\sqrt{6}}\left|0\right\rangle\right)\left(\beta\sqrt{\frac{2}{3}}\left\langle 1\right| + \frac{\alpha}{\sqrt{6}}\left\langle 0\right|\right) + 0 \qquad (3) \\
&= \left(\frac{2\alpha^2}{3} + \frac{\beta^2}{6} + \frac{\alpha^2}{6}\right)\left|0\right\rangle\left\langle 0\right| + \left(\frac{\alpha\beta}{3} + \frac{\alpha\beta}{3}\right)\left|0\right\rangle\left\langle 1\right| \\
&\quad + \left(\frac{\alpha\beta}{3} + \frac{\alpha\beta}{3}\right)\left|1\right\rangle\left\langle 0\right| + \left(\frac{\beta^2}{6} + \frac{\alpha^2}{6} + \frac{2\beta^2}{3}\right)\left|1\right\rangle\left\langle 1\right| \\
&= \frac{2}{3}\left(\alpha\left|0\right\rangle + \beta\left|1\right\rangle\right)\left(\alpha\left\langle 0\right| + \beta\left\langle 1\right|\right) + \frac{\alpha^2+\beta^2}{6}\left(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\right) \\
&= \frac{2}{3}\left|\Psi\right\rangle\left\langle\Psi\right| + \frac{1}{6}\mathbb{1} \qquad (\text{as } \alpha^2+\beta^2=1)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}_2 &= \operatorname{tr}_1(\hat{\rho}_{12}) \\
&= \left(\alpha\sqrt{\frac{2}{3}}\left|0\right\rangle + \frac{\beta}{\sqrt{6}}\left|1\right\rangle\right)\left(\alpha\sqrt{\frac{2}{3}}\left\langle 0\right| + \frac{\beta}{\sqrt{6}}\left\langle 1\right|\right) + \frac{\beta}{\sqrt{6}}\left|0\right\rangle\frac{\beta}{\sqrt{6}}\left\langle 0\right| + 0 \\
&\quad + \frac{\alpha}{\sqrt{6}}\left|1\right\rangle\frac{\alpha}{\sqrt{6}}\left\langle 1\right| + \left(\beta\sqrt{\frac{2}{3}}\left|1\right\rangle + \frac{\alpha}{\sqrt{6}}\left|0\right\rangle\right)\left(\beta\sqrt{\frac{2}{3}}\left\langle 1\right| + \frac{\alpha}{\sqrt{6}}\left\langle 0\right|\right) + 0 \\
&= \hat{\rho}_1 \qquad (\text{from Equation 3})
\end{aligned}
$$

$$
\implies \hat{\rho}_1 = \hat{\rho}_2 = \frac{2}{3}\left|\Psi\right\rangle\left\langle\Psi\right| + \frac{1}{6}\mathbb{1} \equiv \hat{\rho}_{1(2)}
$$

(d)

$$
\begin{aligned}
f_{\mathrm{qc}} &= \left\langle\Psi\right|\hat{\rho}_{1(2)}\left|\Psi\right\rangle \\
&= \left\langle\Psi\right|\left(\frac{2}{3}\left|\Psi\right\rangle\left\langle\Psi\right| + \frac{1}{6}\mathbb{1}\right)\left|\Psi\right\rangle \\
&= \frac{2}{3}\left\langle\Psi\,|\,\Psi\right\rangle\left\langle\Psi\,|\,\Psi\right\rangle + \frac{1}{6}\left\langle\Psi\,|\,\Psi\right\rangle \\
f_{\mathrm{qc}} &= \frac{5}{6}
\end{aligned}
$$

3 Teleportation and the state swapping circuit

(a)

	$\left|\Psi\right\rangle\otimes\left|0\right\rangle$
Apply $\mathbb{1}\otimes\left|0\right\rangle\left\langle 0\right| + \hat{\mathrm{X}}\otimes\left|1\right\rangle\left\langle 1\right|$	$\rightarrow \left|\Psi\right\rangle\otimes\left|0\right\rangle + 0$
Apply $\left|0\right\rangle\left\langle 0\right|\otimes\mathbb{1} + \left|1\right\rangle\left\langle 1\right|\otimes\hat{\mathrm{X}}$	$\rightarrow \left\langle 0\,|\,\Psi\right\rangle\left|0\right\rangle\otimes\left|0\right\rangle + \left\langle 1\,|\,\Psi\right\rangle\left|1\right\rangle\otimes\left|1\right\rangle$
Apply $\mathbb{1}\otimes\left|0\right\rangle\left\langle 0\right| + \hat{\mathrm{X}}\otimes\left|1\right\rangle\left\langle 1\right|$	$\rightarrow \left\langle 0\,|\,\Psi\right\rangle\left|0\right\rangle\otimes\left|0\right\rangle + \left\langle 1\,|\,\Psi\right\rangle\left|0\right\rangle\otimes\left|1\right\rangle$

Thus if $\left|\Phi\right\rangle = \left|0\right\rangle$ the first gate is redundant and can be removed, while the other two cannot.

(b)

Denote $|a\rangle = a_0|0\rangle + a_1|1\rangle$.

$$\begin{aligned}
&& &\text{LHS: } |x\rangle\otimes|y\rangle\otimes|z\rangle \\
&\text{Apply } |0\rangle\langle 0|\otimes\mathbb{1}\otimes\mathbb{1} + |1\rangle\langle 1|\otimes\mathbb{1}\otimes\hat{\mathrm{X}} && \to \langle 0\,|\,x\rangle\,|0\rangle\otimes|y\rangle\otimes|z\rangle + \langle 1\,|\,x\rangle\,|1\rangle\otimes|y\rangle\otimes\hat{\mathrm{X}}\,|z\rangle \\
&& &= x_0|0\rangle\otimes|y\rangle\otimes|z\rangle + x_1|1\rangle\otimes|y\rangle\otimes(z_1|0\rangle + z_0|1\rangle)
\end{aligned}$$

$$\begin{aligned}
&& &\text{RHS: } |x\rangle\otimes|y\rangle\otimes|z\rangle \\
&\text{Apply } \left(|0\rangle\langle 0|\otimes\mathbb{1} + |1\rangle\langle 1|\otimes\hat{\mathrm{X}}\right)\otimes\mathbb{1} && \to \left(\langle 0\,|\,x\rangle\,|0\rangle\otimes|y\rangle + \langle 1\,|\,x\rangle\,|1\rangle\otimes\hat{\mathrm{X}}\,|y\rangle\right)\otimes|z\rangle \\
&& &= [x_0|0\rangle\otimes|y\rangle + x_1|1\rangle\otimes(y_1|0\rangle + y_0|1\rangle)]\otimes|z\rangle \\
&\text{Apply } \mathbb{1}\otimes\left(|0\rangle\langle 0|\otimes\mathbb{1} + |1\rangle\langle 1|\otimes\hat{\mathrm{X}}\right) && \to (x_0|0\rangle\otimes\langle y\,|\,0\rangle\,|0\rangle + x_1y_1|1\rangle\otimes|0\rangle)\otimes|z\rangle \\
&& &\quad + [x_0|0\rangle\otimes\langle y\,|\,1\rangle\,|1\rangle + x_1y_0|1\rangle\otimes|1\rangle]\otimes\hat{\mathrm{X}}\,|z\rangle \\
&& &= (x_0y_0|0\rangle\otimes|0\rangle + x_1y_1|1\rangle\otimes|0\rangle)\otimes|z\rangle \\
&& &\quad + (x_0y_1|0\rangle\otimes|1\rangle + x_1y_0|1\rangle\otimes|1\rangle)\otimes(z_1|0\rangle + z_0|1\rangle) \\
&\text{Apply } \left(|0\rangle\langle 0|\otimes\mathbb{1} + |1\rangle\langle 1|\otimes\hat{\mathrm{X}}\right)\otimes\mathbb{1} && \to x_0y_0|0\rangle\otimes|0\rangle\otimes|z\rangle + x_0y_1|0\rangle\otimes|1\rangle\otimes(z_1|0\rangle + z_0|1\rangle) \\
&& &\quad + x_1y_1|1\rangle\otimes|1\rangle\otimes|z\rangle + x_1y_0|1\rangle\otimes|0\rangle\otimes(z_1|0\rangle + z_0|1\rangle) \\
&\text{Apply } \mathbb{1}\otimes\left(|0\rangle\langle 0|\otimes\mathbb{1} + |1\rangle\langle 1|\otimes\hat{\mathrm{X}}\right) && \to x_0y_0|0\rangle\otimes|0\rangle\,|z\rangle + x_1y_0|1\rangle\otimes|0\rangle\otimes(z_1|0\rangle + z_0|1\rangle) \\
&& &\quad + x_0y_1|0\rangle\otimes|1\rangle\otimes(z_1|1\rangle + z_0|0\rangle) + x_1y_1|1\rangle\otimes|1\rangle\otimes(z_1|0\rangle + z_0|1\rangle) \\
&& &= x_0|0\rangle\otimes(y_0|0\rangle + y_1|1\rangle)\otimes|z\rangle + x_1|1\rangle\otimes(y_0|0\rangle + y_1|1\rangle)\otimes(z_1|0\rangle + z_0|1\rangle) \\
&& &= x_0|0\rangle\otimes|y\rangle\otimes|z\rangle + x_1|1\rangle\otimes|y\rangle\otimes(z_1|0\rangle + z_0|1\rangle) \\
&& &= \left[|0\rangle\langle 0|\otimes\mathbb{1}\otimes\mathbb{1} + |1\rangle\langle 1|\otimes\mathbb{1}\otimes\hat{\mathrm{X}}\right][|x\rangle\otimes|y\rangle\otimes|z\rangle]
\end{aligned}$$

Thus the diagrams are equivalent.

(c)

$$\begin{aligned}
\left[\hat{\mathrm{H}}\otimes\mathbb{1}\right]\left[\mathbb{1}\otimes|0\rangle\langle 0| + \hat{\mathrm{Z}}\otimes|1\rangle\langle 1|\right]\left[\hat{\mathrm{H}}\otimes\mathbb{1}\right] &= \hat{\mathrm{H}}\mathbb{1}\hat{\mathrm{H}}\otimes\mathbb{1}\,|0\rangle\langle 0|\,\mathbb{1} + \hat{\mathrm{H}}\hat{\mathrm{Z}}\hat{\mathrm{H}}\otimes\mathbb{1}\,|1\rangle\langle 1|\,\mathbb{1} \\
&= \mathbb{1}\otimes|0\rangle\langle 0| + \hat{\mathrm{X}}\otimes|1\rangle\langle 1| \qquad (\text{as } \hat{\mathrm{H}}^2 = \mathbb{1},\ \hat{\mathrm{H}}\hat{\mathrm{Z}}\hat{\mathrm{H}} = \hat{\mathrm{X}})
\end{aligned}$$

$$
\begin{aligned}
\left[\hat{\mathrm{H}}\otimes\mathbb{1}\right]\left[|0\rangle\langle 0|\otimes\mathbb{1}+|1\rangle\langle 1|\otimes\hat{\mathrm{Z}}\right]\left[\hat{\mathrm{H}}\otimes\mathbb{1}\right] &= \hat{\mathrm{H}}\,|0\rangle\langle 0|\,\hat{\mathrm{H}}\otimes\mathbb{1}\mathbb{1}\mathbb{1}+\hat{\mathrm{H}}\,|1\rangle\langle 1|\,\hat{\mathrm{H}}\otimes\mathbb{1}\hat{\mathrm{Z}}\mathbb{1} \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)\frac{1}{\sqrt{2}}\left(\langle 0|+\langle 1|\right)\otimes\mathbb{1} \\
&\quad+\frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right)\frac{1}{\sqrt{2}}\left(\langle 0|-\langle 1|\right) \\
&\qquad\qquad\left(\text{as } \hat{\mathrm{H}}\,|x\rangle=\tfrac{1}{\sqrt{2}}\left(|0\rangle+(-1)^x\,|1\rangle\right)\right) \\
&= \frac{1}{2}\left(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|+|1\rangle\langle 1|\right)\otimes\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right) \\
&\quad+\frac{1}{2}\left(|0\rangle\langle 0|-|0\rangle\langle 1|-|1\rangle\langle 0|+|1\rangle\langle 1|\right)\otimes\left(|0\rangle\langle 0|-|1\rangle\langle 1|\right) \\
&\qquad\qquad\left(\text{as } \mathbb{1}=|0\rangle\langle 0|+|1\rangle\langle 1|,\ \hat{\mathrm{Z}}=|0\rangle\langle 0|-|1\rangle\langle 1|\right) \\
&= \frac{1}{2}\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)\otimes\left(|0\rangle\langle 0|+|1\rangle\langle 1|+|0\rangle\langle 0|-|1\rangle\langle 1|\right) \\
&\quad+\frac{1}{2}\left(|0\rangle\langle 1|+|1\rangle\langle 0|\right)\otimes\left(|0\rangle\langle 0|+|1\rangle\langle 1|-|0\rangle\langle 0|+|1\rangle\langle 1|\right) \\
&= \left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)\otimes|0\rangle\langle 0|+\left(|0\rangle\langle 1|+|1\rangle\langle 0|\right)\otimes|1\rangle\langle 1| \\
&= \mathbb{1}\otimes|0\rangle\langle 0|+\hat{\mathrm{X}}\otimes|1\rangle\langle 1| \\
&\qquad\qquad\left(\text{as } \mathbb{1}=|0\rangle\langle 0|+|1\rangle\langle 1|,\ \hat{\mathrm{X}}=|0\rangle\langle 1|+|1\rangle\langle 0|\right)
\end{aligned}
$$

Thus the last two diagrams are equivalent to the first.

(d)

From (b) we have that the first four gates are equivalent to $|0\rangle\langle 0|\otimes\mathbb{1}\otimes\mathbb{1}+|1\rangle\langle 1|\otimes\mathbb{1}\otimes\hat{\mathrm{X}}$. From (c) we have that the last two gates are equivalent to $\mathbb{1}\otimes\mathbb{1}\otimes|0\rangle\langle 0|+\hat{\mathrm{X}}\otimes\mathbb{1}\otimes|1\rangle\langle 1|$. Thus the entire circuit is equivalent to

$$
\begin{aligned}
&\left(\mathbb{1}\otimes\mathbb{1}\otimes|0\rangle\langle 0|+\hat{\mathrm{X}}\otimes\mathbb{1}\otimes|1\rangle\langle 1|\right)\left(|0\rangle\langle 0|\otimes\mathbb{1}\otimes\mathbb{1}+|1\rangle\langle 1|\otimes\mathbb{1}\otimes\hat{\mathrm{X}}\right) \\
&=|0\rangle\langle 0|\otimes\mathbb{1}\otimes|0\rangle\langle 0|+|1\rangle\langle 1|\otimes\mathbb{1}\otimes|0\rangle\langle 0|\,\hat{\mathrm{X}}+\hat{\mathrm{X}}\,|0\rangle\langle 0|\otimes\mathbb{1}\otimes|1\rangle\langle 1|+\hat{\mathrm{X}}\,|1\rangle\langle 1|\otimes\mathbb{1}\otimes|1\rangle\langle 1|\,\hat{\mathrm{X}} \\
&=|0\rangle\langle 0|\otimes\mathbb{1}\otimes|0\rangle\langle 0|+|1\rangle\langle 1|\otimes\mathbb{1}\otimes|0\rangle\langle 1|+|1\rangle\langle 0|\otimes\mathbb{1}\otimes|1\rangle\langle 1|+|0\rangle\langle 1|\otimes\mathbb{1}\otimes|1\rangle\langle 0|\,.
\end{aligned}
$$

Applying this to $|\psi\rangle\otimes|\chi\rangle\otimes|0\rangle$ then gives

$$
\begin{aligned}
\langle 0\,|\,\psi\rangle\,|0\rangle\otimes|\chi\rangle\otimes|0\rangle+0+0+\langle 1\,|\,\psi\rangle\,|0\rangle\otimes|\chi\rangle\otimes|1\rangle &= |0\rangle\otimes|\chi\rangle\otimes\left(\psi_0\,|0\rangle+\psi_1\,|1\rangle\,|1\rangle\right) \\
&= |0\rangle\otimes|\chi\rangle\otimes|\psi\rangle.
\end{aligned}
$$

(e)

(1)

$$
\begin{aligned}
& && |\psi\rangle \otimes |0\rangle \otimes |0\rangle \\
&\text{Apply } \mathbb{1} \otimes \hat{\mathrm{H}} \otimes \mathbb{1} && \to |\psi\rangle \otimes \hat{\mathrm{H}}\,|0\rangle \otimes |0\rangle \\
& && = \frac{1}{\sqrt{2}}\,|\psi\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle \\
&\text{Apply } \mathbb{1} \otimes \Big(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \hat{\mathrm{X}}\Big) && \to \frac{1}{\sqrt{2}}\,|\psi\rangle \otimes (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \\
&\text{Apply } \Big(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \hat{\mathrm{X}}\Big) \otimes \mathbb{1} && \to \frac{1}{\sqrt{2}}\,\langle 0\,|\,\psi\rangle\,|0\rangle \otimes (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \\
& && \quad + \frac{1}{\sqrt{2}}\,\langle 1\,|\,\psi\rangle\,|1\rangle \otimes (|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle) \\
& && = \frac{\psi_0}{\sqrt{2}}\,|0\rangle \otimes (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \\
& && \quad + \frac{\psi_1}{\sqrt{2}}\,|1\rangle \otimes (|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle) \\
&\text{Apply } \hat{\mathrm{H}} \otimes \mathbb{1} \otimes \hat{\mathrm{H}} && \to \frac{\psi_0}{2}\,(|0\rangle + |1\rangle) \otimes (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \\
& && \quad + \frac{\psi_1}{2}\,(|0\rangle - |1\rangle) \otimes (|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle) \\
&\text{Apply } \mathbb{1} \otimes \Big(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \hat{\mathrm{X}}\Big) && \to \frac{\psi_0}{2}\,(|0\rangle + |1\rangle) \otimes |0\rangle \otimes |0\rangle + \frac{\psi_1}{2}\,(|0\rangle - |1\rangle) \otimes |0\rangle \otimes |1\rangle \\
& && \quad + \frac{\psi_0}{2}\,(|0\rangle + |1\rangle) \otimes |1\rangle \otimes |0\rangle + \frac{\psi_1}{2}\,(|0\rangle - |1\rangle) \otimes |1\rangle \otimes |1\rangle \\
& && = \frac{\psi_0}{2}\,(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes |0\rangle \\
& && \quad + \frac{\psi_1}{2}\,(|0\rangle - |1\rangle) \otimes ||0\rangle + |1\rangle\rangle \otimes |1\rangle \\
&\text{Apply } |0\rangle\langle 0| \otimes \mathbb{1} \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathbb{1} \otimes \hat{\mathrm{Z}} && \to \frac{\psi_0}{2}\,|0\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle + \frac{\psi_1}{2}\,|0\rangle \otimes (|0\rangle + |1\rangle) \otimes |1\rangle \\
& && \quad + \frac{\psi_0}{2}\,|1\rangle \otimes (|0\rangle + |1\rangle) \otimes \hat{\mathrm{Z}}\,|0\rangle - \frac{\psi_1}{2}\,|1\rangle \otimes (|0\rangle + |1\rangle) \otimes \hat{\mathrm{Z}}\,|1\rangle \\
& && = \frac{\psi_0}{2}\,|0\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle + \frac{\psi_1}{2}\,|0\rangle \otimes (|0\rangle + |1\rangle) \otimes |1\rangle \\
& && \quad + \frac{\psi_0}{2}\,|1\rangle \otimes (|0\rangle + |1\rangle) \otimes |0\rangle + \frac{\psi_1}{2}\,|1\rangle \otimes (|0\rangle + |1\rangle) \otimes |1\rangle \\
& && = \frac{1}{\sqrt{2}}\,(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}\,(|0\rangle + |1\rangle) \otimes (\psi_0\,|0\rangle + \psi_1\,|1\rangle) \\
& && = |+\rangle \otimes |+\rangle \otimes |\psi\rangle
\end{aligned}
$$

(2)

Following the first (Hadamard) gate, the system is in the state $\frac{1}{\sqrt{2}}\psi \otimes (|0\rangle + |1\rangle) \otimes |0\rangle = |\psi\rangle \otimes |+\rangle \otimes |0\rangle$, and so the ancilla and target are in the states $|+\rangle$ and $|0\rangle$, respectively.

After the second (C-NOT) gate (just before the red line), the system is in the state $\frac{1}{\sqrt{2}}\,|\psi\rangle \otimes (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$, and so the ancilla and target are in the entangled state $\frac{1}{\sqrt{2}}\,(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) = |\Psi_{00}\rangle$.

(3)

Alice measuring her two qubits corresponds to $|i\rangle\langle i| \otimes |j\rangle\langle j|$ acting on her two quibits. In terms of the whole system, this is given by $|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| \otimes \mathbb{1}$ acting on the system. Acting this on $|+\rangle \otimes |+\rangle \otimes |\psi\rangle$ then gives $\langle\alpha\,|\,+\rangle\,|\alpha\rangle \otimes \langle\beta\,|\,+\rangle\,|\beta\rangle \otimes |\psi\rangle$. Regardless of the choice of $\alpha, \beta = 0, 1$ (z-basis $\implies$ $|0\rangle, |1\rangle$), Bob's state $|\psi\rangle$ is unchanged. This is due to the fact that Bob's target state is no longer entangled with any other state, and so only measurements on Bob's state can affect it.

(f)

$$\begin{aligned}
(|i\rangle\langle i| \otimes \mathbb{1})\Big(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \hat{\mathrm{U}}\Big)\left[(\alpha|0\rangle + \beta|1\rangle) \otimes |\psi\rangle\right] &= (|i\rangle\langle i| \otimes \mathbb{1})\Big(\alpha|0\rangle \otimes |\psi\rangle + \beta|1\rangle \otimes \hat{\mathrm{U}}|\psi\rangle\Big) \\
&= \alpha\langle i\,|\,0\rangle\,|i\rangle \otimes |\psi\rangle + \beta\langle i\,|\,1\rangle\,|i\rangle \otimes \hat{\mathrm{U}}|\psi\rangle \\
\Big(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \hat{\mathrm{U}}\Big)(|i\rangle\langle i| \otimes \mathbb{1})\left[(\alpha|0\rangle + \beta|1\rangle) \otimes |\psi\rangle\right] &= \Big(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \hat{\mathrm{U}}\Big)\left[(\alpha\langle i\,|\,0\rangle + \beta\langle i\,|\,1\rangle)\,|i\rangle \otimes |\psi\rangle\right] \\
&= \alpha\langle i\,|\,0\rangle\langle 0\,|\,i\rangle\,|0\rangle \otimes |\psi\rangle + \beta\langle i\,|\,1\rangle\langle 1\,|\,i\rangle\,|1\rangle \otimes \hat{\mathrm{U}}|\psi\rangle
\end{aligned}$$

Thus in both cases the probability of measuring $|0\rangle$ and $|1\rangle$ is $|\alpha|^2$ and $|\beta|^2$ respectively, and so the circuits are equivalent.

In the circuit in (e), the final two gates are controlled unitary gates, with Alice's qubit and the ancilla qubit acting as the controls. Therefore Alice measuring her qubit or the ancilla qubit at the end of the circuit is equivalent to measuring them before the last two gates. Before these two gates, the system is in the state

$$\begin{aligned}
&\frac{\psi_0}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) + \frac{\psi_1}{2}(|0\rangle - |1\rangle) \otimes (|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle) \\
&= \frac{1}{2}\left[|00\rangle(\psi_0|0\rangle + \psi_1|1\rangle) + |01\rangle(\psi_0|1\rangle + \psi_1|0\rangle) + |10\rangle(\psi_0|0\rangle - \psi_1|1\rangle) + |11\rangle(\psi_0|1\rangle - \psi_1|0\rangle)\right].
\end{aligned}$$

Therefore Bob's state collapses into one of four states once Alice measures one of $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$. If Alice communicates the results of measuring her qubit and the ancilla qubit to Bob, then Bob can apply the appropriate gate to his qubit to obtain $|\psi\rangle$. In particular,

$$\begin{aligned}
&\text{Alice measuring } |00\rangle \to \psi_0|0\rangle + \psi_1|1\rangle && \text{Bob applying } \mathbb{1} \to \psi_0|0\rangle + \psi_1|1\rangle = |\psi\rangle \\
&\text{Alice measuring } |01\rangle \to \psi_0|1\rangle + \psi_1|0\rangle && \text{Bob applying } \hat{\mathrm{X}} \to \psi_0|0\rangle + \psi_1|1\rangle = |\psi\rangle \\
&\text{Alice measuring } |10\rangle \to \psi_0|0\rangle - \psi_1|1\rangle && \text{Bob applying } \hat{\mathrm{Z}} \to \psi_0|0\rangle + \psi_1|1\rangle = |\psi\rangle \\
&\text{Alice measuring } |11\rangle \to \psi_0|1\rangle - \psi_1|0\rangle && \text{Bob applying } \hat{\mathrm{Z}}\hat{\mathrm{X}} \to \psi_0|0\rangle + \psi_1|1\rangle = |\psi\rangle
\end{aligned}$$

Thus Alice's qubit $|\psi\rangle$ can be teleported to Bob by replacing the last two gates in the circuit with measurements on Alice's qubit and the ancilla qubit, classical communication of these measurements to Bob, and a gate on Bob's qubit depending on the result of these measurements.