PYU44T20 Quantum Optics and Information
Problem Set 1 due 14/02/2023

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SS Theoretical Physics

1 Linear Vector Spaces & Operators

1.1)

For a given Hermitian operator $\hat{A}$ we have

$$\hat{A} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} \alpha + \beta_z & \beta_x - i\beta_y \\ \beta_x + i\beta_y & \alpha - \beta_z \end{pmatrix}.$$

This is equivalent to

$$a_{00} = \alpha + \beta_z, \tag{1}$$
$$a_{11} = \alpha - \beta_z, \tag{2}$$
$$a_{01} = \beta_x - i\beta_y, \tag{3}$$
$$a_{10} = \beta_x + i\beta_y. \tag{4}$$

These can be rearranged to obtain expressions for α and $\vec{\beta}$ as

$$(1) + (2) \implies \alpha = \frac{1}{2}\left(a_{00} + a_{11}\right), \tag{5}$$
$$(1) - (2) \implies \beta_z = \frac{1}{2}\left(a_{00} - a_{11}\right), \tag{6}$$
$$(3) + (4) \implies \beta_x = \frac{1}{2}\left(a_{01} + a_{10}\right), \tag{7}$$
$$(3) - (4) \implies \beta_y = \frac{i}{2}\left(a_{01} - a_{10}\right). \tag{8}$$

If $\hat{A} = \hat{0}$, i.e. $a_{00} = a_{01} = a_{10} = a_{11} = 0$, then we also have $\alpha = \beta_x = \beta_y = \beta_z = 0$, and so the elements of $\left\{\hat{\mathbb{1}}, \hat{\sigma}_i\right\}$ are linearly independent.

For general $\hat{A}$, from $\hat{A}^\dagger = \hat{A}$ we have

$$a_{00}^* = a_{00} \implies a_{00} \in \mathbb{R}, \tag{9}$$
$$a_{11}^* = a_{11} \implies a_{11} \in \mathbb{R}, \tag{10}$$
$$a_{01} = a_{10}^*. \tag{11}$$

Combining these constraints with the expressions for $\alpha, \vec{\beta}$ gives

$$(5), (9), (10) \implies \alpha \in \mathbb{R}$$
$$(6), (9), (10) \implies \beta_z \in \mathbb{R}$$
$$(7), (11) \implies \beta_x^* = \beta_x \implies \beta_x \in \mathbb{R}$$
$$(8), (11) \implies \beta_y^* = \beta_y \implies \beta_y \in \mathbb{R}$$

Thus $\left\{\hat{\mathbb{1}}, \hat{\sigma}_i\right\}$ spans the space of Hermitian operators, and so any $\hat{A}$ can be expressed in the desired form $\hat{A} = \alpha\hat{\mathbb{1}} + \vec{\beta} \cdot \vec{\hat{\sigma}}$.

1.2)

$$\begin{aligned}
\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)^2 &= \left(\beta_x\hat{\sigma}_x+\beta_y\hat{\sigma}_y+\beta_z\hat{\sigma}_z\right)^2\\
&= \left(\beta_x^2\hat{\sigma}_x\hat{\sigma}_x+\beta_y^2\hat{\sigma}_y\hat{\sigma}_y+\beta_z^2\hat{\sigma}_z\hat{\sigma}_z\right)\\
&\quad+\beta_x\beta_y\left(\hat{\sigma}_x\hat{\sigma}_y+\hat{\sigma}_y\hat{\sigma}_x\right)+\beta_x\beta_z\left(\hat{\sigma}_x\hat{\sigma}_z+\hat{\sigma}_z\hat{\sigma}_x\right)+\beta_y\beta_z\left(\hat{\sigma}_y\hat{\sigma}_z+\hat{\sigma}_z\hat{\sigma}_y\right)\\
&= \left(\frac{\beta_x^2}{2}\left\{\hat{\sigma}_x,\hat{\sigma}_x\right\}+\frac{\beta_y^2}{2}\left\{\hat{\sigma}_y,\hat{\sigma}_y\right\}+\frac{\beta_z^2}{2}\left\{\hat{\sigma}_z,\hat{\sigma}_z\right\}\right)\\
&\quad+\beta_x\beta_y\left\{\hat{\sigma}_x,\hat{\sigma}_y\right\}+\beta_x\beta_z\left\{\hat{\sigma}_x,\hat{\sigma}_z\right\}+\beta_y\beta_z\left\{\hat{\sigma}_y,\hat{\sigma}_z\right\}\\
&= \left(\beta_x^2+\beta_y^2+\beta_z^2\right)\hat{\mathbb{1}}+0 \qquad \text{(from } \left\{\hat{\sigma}_i,\hat{\sigma}_j\right\}=2\delta_{ij}\hat{\mathbb{1}}\text{)}
\end{aligned}$$

1.3)

$$\begin{aligned}
\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)^{2n} &= \left[\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)^2\right]^n\\
&= \left[\left(\vec{\beta}\cdot\vec{\beta}\right)\hat{\mathbb{1}}\right]^n\\
&= \left(\vec{\beta}\cdot\vec{\beta}\right)^n\hat{\mathbb{1}}
\end{aligned}$$

$$\begin{aligned}
\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)^{2n+1} &= \left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)^{2n}\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)\\
&= \left(\vec{\beta}\cdot\vec{\beta}\right)^n\hat{\mathbb{1}}\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)\\
&= \left(\vec{\beta}\cdot\vec{\beta}\right)^n\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)
\end{aligned}$$

1.4)

$$\hat{H}_0 = \begin{pmatrix} E & K \\ K^* & E \end{pmatrix} = \alpha\hat{\mathbb{1}} + \vec{\beta}\cdot\hat{\vec{\sigma}}$$

$$(5),(6),(7),(8) \implies \alpha = E, \quad \beta_x = \frac{K+K^*}{2} = \mathrm{Re}(K), \quad \beta_y = \frac{i\,(K-K^*)}{2} = -\,\mathrm{Im}(K), \quad \beta_z = 0$$

$$\begin{aligned}
\hat{U}(t) &= \exp\left(-\frac{i\hat{H}_0 t}{\hbar}\right) \\
&= \exp\left(-\frac{it}{\hbar}\left(\alpha\hat{\mathbb{1}} + \vec{\beta}\cdot\hat{\vec{\sigma}}\right)\right) \\
&= \exp\left(-\frac{it\alpha}{\hbar}\hat{\mathbb{1}}\right)\exp\left(-\frac{it}{\hbar}\vec{\beta}\cdot\hat{\vec{\sigma}}\right) \\
&= \exp\left(-\frac{it\alpha}{\hbar}\hat{\mathbb{1}}\right)\sum_{k=0}^{\infty}\frac{\left(-\frac{it}{\hbar}\right)^k}{k!}\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)^k \\
&= \exp\left(-\frac{iEt}{\hbar}\hat{\mathbb{1}}\right)\sum_{n=0}^{\infty}\left[\frac{\left(-\frac{it}{\hbar}\right)^{2n}}{(2n)!}\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)^{2n} + \frac{\left(-\frac{it}{\hbar}\right)^{2n+1}}{(2n+1)!}\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)^{2n+1}\right] \\
&= \exp\left(-\frac{iEt}{\hbar}\hat{\mathbb{1}}\right)\sum_{n=0}^{\infty}\left[\frac{(-1)^n\left(\frac{t}{\hbar}\right)^{2n}}{(2n)!}\left(\beta_x^2+\beta_y^2+\beta_z^2\right)^n\hat{\mathbb{1}} - i\frac{(-1)^n\left(\frac{t}{\hbar}\right)^{2n+1}}{(2n+1)!}\left(\beta_x^2+\beta_y^2+\beta_z^2\right)^n\left(\vec{\beta}\cdot\hat{\vec{\sigma}}\right)\right] \\
&= \exp\left(-\frac{iEt}{\hbar}\hat{\mathbb{1}}\right)\sum_{n=0}^{\infty}\left[\frac{(-1)^n\left(\frac{t}{\hbar}\sqrt{\mathrm{Re}(K)^2+\left(-\,\mathrm{Im}(K)\right)^2}\right)^{2n}}{(2n)!}\hat{\mathbb{1}}\right. \\
&\qquad\left. - \frac{i(-1)^n\left(\frac{t}{\hbar}\sqrt{\mathrm{Re}(K)^2+\left(-\,\mathrm{Im}(K)\right)^2}\right)^{2n+1}}{(2n+1)!\sqrt{\mathrm{Re}(K)^2+\left(-\,\mathrm{Im}(K)\right)^2}}\left(\beta_x\hat{\sigma}_x+\beta_y\hat{\sigma}_y\right)\right] \\
&= \exp\left(-\frac{iEt}{\hbar}\hat{\mathbb{1}}\right)\left[\cos\left(\frac{|K|t}{\hbar}\right)\hat{\mathbb{1}} - \frac{i}{|K|}\sin\left(\frac{|K|t}{\hbar}\right)\left(\beta_x\hat{\sigma}_x+\beta_y\hat{\sigma}_y\right)\right] \\
&= \exp\left(-\frac{iEt}{\hbar}\right)\left[\cos\left(\frac{|K|t}{\hbar}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix} - \frac{i\sin\left(\frac{|K|t}{\hbar}\right)}{|K|}\left(\mathrm{Re}(K)\begin{pmatrix}0&1\\1&0\end{pmatrix} + i\,\mathrm{Im}(K)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\right] \\
&= \exp\left(-\frac{iEt}{\hbar}\right)\begin{pmatrix}\cos\left(\frac{|K|t}{\hbar}\right) & -i\frac{K}{|K|}\sin\left(\frac{|K|t}{\hbar}\right) \\ -i\frac{K^*}{|K|}\sin\left(\frac{|K|t}{\hbar}\right) & \cos\left(\frac{|K|t}{\hbar}\right)\end{pmatrix}
\end{aligned}$$

1.5)

$$
\begin{aligned}
0 &= \det\left(\hat{H}_0 - \lambda\hat{\mathbb{1}}\right) \\
&= \begin{vmatrix} E-\lambda & K \\ K^* & E-\lambda \end{vmatrix} \\
&= (E-\lambda)^2 - |K|^2 \\
\implies \lambda_\pm &= E \pm |K|
\end{aligned}
$$

$$
\hat{H}_0 |v_\pm\rangle = \lambda_\pm |v_\pm\rangle \implies \begin{pmatrix} E & K \\ K^* & E \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} aE + bK \\ aK^* + bE \end{pmatrix} = \begin{pmatrix} a(E \pm |K|) \\ b(E \pm |K|) \end{pmatrix}
$$

$$
\implies aE + bK = a(E \pm |K|), \qquad aK^* + bE = b(E \pm |K|)
$$

$$
\implies b = \pm a\frac{|K|}{K} = \pm a\frac{K^*}{|K|} \implies |v_\pm\rangle = a\left(|0\rangle \pm \frac{|K|}{K}|1\rangle\right)
$$

$$
\begin{aligned}
1 &= \langle v_\pm \,|\, v_\pm\rangle \\
&= |a|^2\left(1^2 + \frac{|K|^2}{|K|^2}\right) \\
\implies |a|^2 &= \frac{1}{2} \\
\implies |v_\pm\rangle &= \frac{e^{i\phi}}{\sqrt{2}}\left(|0\rangle \pm \frac{|K|}{K}|1\rangle\right),\ \phi \in \mathbb{R}
\end{aligned}
$$

We can express $\hat{H}_0$ as a matrix in the standard basis or the eigenbasis, i.e.

$$
\begin{aligned}
\hat{H}_0 &= E\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) + K|0\rangle\langle 1| + K^*|1\rangle\langle 0| \\
&= \begin{pmatrix} E & K \\ K^* & E \end{pmatrix}_{|0\rangle,|1\rangle}
\end{aligned}
\qquad
\begin{aligned}
\hat{H}_0 &= \lambda_+|v_+\rangle\langle v_+| + \lambda_-|v_-\rangle\langle v_-| \\
&= \begin{pmatrix} E+|K| & 0 \\ 0 & E-|K| \end{pmatrix}_{|v_\pm\rangle}
\end{aligned}
$$

The corresponding form of $\hat{U}(t)$ as a matrix in the eigenbasis is thus given by

$$
\begin{aligned}
\hat{U}(t) &= \exp\left(-\frac{i\hat{H}_0 t}{\hbar}\right) \\
&= \exp\left(-\frac{it}{\hbar}\,\mathrm{diag}(E+|K|, E-|K|)_{|v_\pm\rangle}\right) \\
\hat{U}(t) &= \mathrm{diag}\left(\exp\left(-\frac{it}{\hbar}(E+|K|)\right), \exp\left(-\frac{it}{\hbar}(E-|K|)\right)\right)_{|v_\pm\rangle} \\
&= \begin{pmatrix} \exp\left(-\frac{it}{\hbar}(E+|K|)\right) & 0 \\ 0 & \exp\left(-\frac{it}{\hbar}(E-|K|)\right) \end{pmatrix}_{|v_\pm\rangle}
\end{aligned}
$$

2 Projection Operators

2.1)

$$\begin{aligned}\hat{P}_i^2 &= (|i\rangle\langle i|)(|i\rangle\langle i|) \\ &= |i\rangle\langle i\,|\,i\rangle\langle i| \\ &= |i\rangle\,\hat{\mathbb{1}}\,\langle i| \\ &= \hat{P}_i\end{aligned} \qquad \begin{aligned}\hat{P}_i^\dagger &= (|i\rangle\langle i|)^\dagger \\ &= (\langle i|)^\dagger(|i\rangle)^\dagger \\ &= |i\rangle\langle i| \\ &= \hat{P}_i\end{aligned}$$

$$\begin{aligned}|i\rangle \text{ not normalised } &\implies \hat{P}_i^2 = \langle i\,|\,i\rangle\,\hat{P}_i \neq \hat{P}_i \\ &\implies \text{first condition } \hat{P}_i^2 = \hat{P}_i \text{ is not satisfied}\end{aligned}$$

2.2)

$$\begin{aligned}\left(\hat{P}_i + \hat{P}_j\right)^2 &= (|i\rangle\langle i| + |j\rangle\langle j|)(|i\rangle\langle i| + |j\rangle\langle j|) \\ &= |i\rangle\langle i\,|\,i\rangle\langle i| + |i\rangle\langle i\,|\,j\rangle\langle j| + |j\rangle\langle j\,|\,i\rangle\langle j| + |j\rangle\langle j\,|\,j\rangle\langle j| \\ &= |i\rangle\langle i| + 0 + 0 + |j\rangle\langle j| \\ &= \hat{P}_i + \hat{P}_j\end{aligned}$$

$$\begin{aligned}\left(\hat{P}_i + \hat{P}_j\right)^\dagger &= (|i\rangle\langle i| + |j\rangle\langle j|)^\dagger \\ &= (|i\rangle\langle i|)^\dagger + (|j\rangle\langle j|)^\dagger \\ &= \hat{P}_i^\dagger + \hat{P}_j^\dagger \\ &= \hat{P}_i + \hat{P}_j\end{aligned}$$

2.3)

$$\begin{aligned}\sum_{i=1}^{\dim\mathcal{H}} \langle C_i|\,\hat{A}\,|C_i\rangle &= \sum_{i=1}^{\dim\mathcal{H}} \langle C_i|\,\hat{A}\left(\sum_{j=1}^{\dim\mathcal{H}} |B_j\rangle\langle B_j\,|\,C_i\rangle\right) \\ &= \sum_{i,j=1}^{\dim\mathcal{H}} \langle C_i|\,\hat{A}\,|B_j\rangle\langle B_j\,|\,C_i\rangle \\ &= \sum_{j=1}^{\dim\mathcal{H}} \left(\sum_{i=1}^{\dim\mathcal{H}} \langle B_j\,|\,C_i\rangle\langle C_i|\right)\hat{A}\,|B_j\rangle \\ &= \sum_{j=1}^{\dim\mathcal{H}} \langle B_j|\,\hat{A}\,|B_j\rangle\end{aligned}$$

2.4)

$$\begin{aligned}\mathrm{Tr}\left[\hat{P}_i\hat{A}\right] &= \sum_{j=1}^{\dim\mathcal{H}} \langle B_j|\,\hat{P}_i\hat{A}\,|B_j\rangle \\ &= \sum_{j=1}^{\dim\mathcal{H}} \langle B_j\mid B_i\rangle\,\langle B_i|\,\hat{A}\,|B_j\rangle \\ &= \sum_{j=1}^{\dim\mathcal{H}} \delta_{ji}\,\langle B_i|\,\hat{A}\,|B_j\rangle \\ &= \langle B_i|\,\hat{A}\,|B_i\rangle\end{aligned}$$

2.5)

$$\begin{aligned}\mathrm{Tr}\left[\hat{A}_1\hat{A}_2\right] &= \sum_{i=1}^{\dim\mathcal{H}} \langle B_i|\,\hat{A}_1\hat{A}_2\,|B_i\rangle \\ &= \sum_{i=1}^{\dim\mathcal{H}} \langle B_i|\,\hat{A}_1\hat{\mathbb{1}}\hat{A}_2\,|B_i\rangle \\ &= \sum_{i=1}^{\dim\mathcal{H}} \langle B_i|\,\hat{A}_1\left(\sum_{j=1}^{\dim\mathcal{H}} |B_j\rangle\,\langle B_j|\right)\hat{A}_2\,|B_i\rangle \\ &= \sum_{i,j=1}^{\dim\mathcal{H}} \langle B_i|\,\hat{A}_1\,|B_j\rangle\,\langle B_j|\,\hat{A}_2\,|B_i\rangle \\ &= \sum_{i,j=1}^{\dim\mathcal{H}} \langle B_j|\,\hat{A}_2\,|B_i\rangle\,\langle B_i|\,\hat{A}_1\,|B_j\rangle \\ &= \sum_{j=1}^{\dim\mathcal{H}} \langle B_j|\,\hat{A}_2\left(\sum_{i=1}^{\dim\mathcal{H}} |B_i\rangle\,\langle B_i|\right)\hat{A}_1\,|B_j\rangle \\ &= \sum_{j=1}^{\dim\mathcal{H}} \langle B_j|\,\hat{A}_2\hat{\mathbb{1}}\hat{A}_1\,|B_j\rangle \\ &= \sum_{j=1}^{\dim\mathcal{H}} \langle B_j|\,\hat{A}_2\hat{A}_1\,|B_j\rangle \\ &= \mathrm{Tr}\left[\hat{A}_2\hat{A}_1\right]\end{aligned}$$

2.6)

$$\begin{aligned}|D_i\rangle &= \hat{\mathbb{1}}\,|D_i\rangle \\ &= \sum_{j=1}^{\dim\mathcal{H}} |D_j\rangle\,\langle D_j\mid D_i\rangle \\ &= \sum_{j=1}^{\dim\mathcal{H}} \langle D_j\mid D_i\rangle\,|D_j\rangle\end{aligned}$$

$$\text{also}\;\; |D_i\rangle = \sum_{j=1}^{\dim\mathcal{H}} \delta_{ji}\,|D_j\rangle \implies \langle D_j\mid D_i\rangle = \delta_{ji} \implies |D_i\rangle\;\text{ are orthonormal}$$

3 Translation Operators

3.1)

$$\begin{aligned}
\left(\prod_{i=1}^{N}\hat{U}_i\right)\left(\prod_{j=1}^{N}\hat{U}_j\right)^\dagger &= \hat{U}_1\cdots\hat{U}_N\hat{U}_N^\dagger\cdots\hat{U}_1^\dagger \\
&= \hat{U}_1\cdots\hat{U}_{N-1}\hat{\mathbb{1}}\hat{U}_{N-1}^\dagger\cdots\hat{U}_1^\dagger \\
&= \hat{U}_1\cdots\hat{U}_{N-1}\hat{U}_{N-1}^\dagger\cdots\hat{U}_1^\dagger \\
&= \dots \\
&= \hat{U}_1\hat{U}_1^\dagger \\
&= \hat{\mathbb{1}} \\
\implies \left(\prod_{i=1}^{N}\hat{U}_i\right)^\dagger &= \left(\prod_{i=1}^{N}\hat{U}_i\right)^{-1} \implies \text{unitary}
\end{aligned}$$

3.2)

$$\begin{aligned}
\left(\hat{U}_1+\hat{U}_2\right)\left(\hat{U}_1+\hat{U}_2\right)^\dagger &= \left(\hat{U}_1+\hat{U}_2\right)\left(\hat{U}_1^\dagger+\hat{U}_2^\dagger\right) \\
&= \hat{U}_1\hat{U}_1^\dagger+\hat{U}_1\hat{U}_2^\dagger+\hat{U}_2\hat{U}_1^\dagger+\hat{U}_2\hat{U}_2^\dagger \\
&= \hat{\mathbb{1}}+\hat{U}_1\hat{U}_2^\dagger+\hat{U}_2\hat{U}_1^\dagger+\hat{\mathbb{1}}
\end{aligned}$$

Thus the sum is only unitary if $\hat{U}_1\hat{U}_2^\dagger+\hat{U}_2\hat{U}_1^\dagger = -\hat{\mathbb{1}}$, which is not generally true for arbitrary unitary $\hat{U}_1, \hat{U}_2$, e.g.

$$\begin{aligned}
\hat{\sigma}_i\hat{\sigma}j^\dagger+\hat{\sigma}_j\hat{\sigma}_i^\dagger &= \hat{\sigma}_i\hat{\sigma}_j+\hat{\sigma}_j\hat{\sigma}_i \\
&= 2\delta_{ij}\hat{\mathbb{1}} \neq -\hat{\mathbb{1}} \\
\hat{U}\hat{U}^\dagger+\hat{U}\hat{U}^\dagger &= 2\hat{\mathbb{1}} \neq -\hat{\mathbb{1}}
\end{aligned}$$

3.3)

$$\begin{aligned}
\left(\hat{T}_1\right)^2 &= \left(\sum_{i=-\infty}^{\infty}\left|x_{i-1}\right\rangle\left\langle x_i\right|\right)\left(\sum_{j=-\infty}^{\infty}\left|x_{j-1}\right\rangle\left\langle x_j\right|\right) \\
&= \sum_{i,j=-\infty}^{\infty}\left|x_{i-1}\right\rangle\left\langle x_i \,\middle|\, x_{j-1}\right\rangle\left\langle x_j\right| \\
&= \sum_{i,j=-\infty}^{\infty}\left|x_{i-1}\right\rangle\delta_{i,j-1}\left\langle x_j\right| \\
&= \sum_{j=-\infty}^{\infty}\left|x_{j-2}\right\rangle\left\langle x_j\right| \\
&= \hat{T}_2
\end{aligned}$$

3.4)

We can prove this by induction, i.e. prove for $k = 1$, and then prove for $k = n + 1$ assuming true for $k = n$.

$$\left(\hat{T}_1\right)^1 = \hat{T}_1 \implies \text{true for } k = 1$$

Now assume this is true for $k = n$.

$$\begin{aligned}
\left(\hat{T}_1\right)^{n+1} &= \left(\hat{T}_1\right)^n \hat{T}_1 \\
&= \hat{T}_n \hat{T}_1 \\
&= \left(\sum_{i=-\infty}^{\infty} |x_{i-n}\rangle \langle x_i|\right) \left(\sum_{j=-\infty}^{\infty} |x_{j-1}\rangle \langle x_j|\right) \\
&= \sum_{i,j=-\infty}^{\infty} |x_{i-n}\rangle \langle x_i \,|\, x_{j-1}\rangle \langle x_j| \\
&= \sum_{i,j=-\infty}^{\infty} |x_{i-n}\rangle \delta_{i,j-1} \langle x_j| \\
&= \sum_{j=-\infty}^{\infty} |x_{j-(n+1)}\rangle \langle x_j| \\
&= \hat{T}_{n+1} \implies \text{true for } k = n+1
\end{aligned}$$

$$\begin{aligned}
\text{True for } k = 1 &\implies \text{true for } k = 2 \\
&\implies \text{true for } k = 3 \\
&\implies \dots \\
&\implies \text{true for } k \in \mathbb{Z}^+
\end{aligned}$$

3.5)

$$\begin{aligned}
\hat{T}_1 \hat{T}_1^\dagger &= \left(\sum_{i=-\infty}^{\infty} |x_{i-1}\rangle \langle x_i|\right) \left(\sum_{j=-\infty}^{\infty} |x_{j-1}\rangle \langle x_j|\right)^\dagger \\
&= \left(\sum_{i=-\infty}^{\infty} |x_{i-1}\rangle \langle x_i|\right) \left(\sum_{j=-\infty}^{\infty} |x_j\rangle \langle x_{j-1}|\right) \\
&= \sum_{i,j=-\infty}^{\infty} |x_{i-1}\rangle \langle x_i \,|\, x_j\rangle \langle x_{j-1}| \\
&= \sum_{i,j=-\infty}^{\infty} |x_{i-1}\rangle \delta_{ij} \langle x_{j-1}| \\
&= \sum_{j=-\infty}^{\infty} |x_{j-1}\rangle \langle x_{j-1}| \\
&= \sum_{k=-\infty}^{\infty} |x_k\rangle \langle x_k| \qquad (k = j-1) \\
&= \hat{\mathbb{1}}
\end{aligned}$$

$$\implies \hat{T}_1^\dagger = \hat{T}_1^{-1} \implies \hat{T}_1 \text{ is unitary}$$

From 3.4), $\hat{T}_k = \left(\hat{T}_1\right)^k$, and so is a product of k unitary operators. From 3.1), this product is also a unitary operator, and thus $\hat{T}_k$ must also be unitary.

3.6)

$$
\begin{aligned}
\hat{T}_p\hat{X}\hat{T}_p^\dagger &= \left(\sum_{i=-\infty}^{\infty} |x_{i-p}\rangle\langle x_i|\right)\left(\sum_{j=-\infty}^{\infty} x_j\,|x_j\rangle\langle x_j|\right)\left(\sum_{k=-\infty}^{\infty} |x_{k-p}\rangle\langle x_k|\right)^\dagger \\
&= \sum_{i,j,k=-\infty}^{\infty} x_j\,|x_{i-p}\rangle\langle x_i \mid x_j\rangle\langle x_j \mid x_k\rangle\langle x_{k-p}| \\
&= \sum_{i,j,k=-\infty}^{\infty} ja\,|x_{i-p}\rangle\,\delta_{ij}\delta_{jk}\,\langle x_{k-p}| \\
&= \sum_{j=-\infty}^{\infty} ja\,|x_{j-p}\rangle\langle x_{j-p}| \\
&= \sum_{l=-\infty}^{\infty} (l+p)\,a\,|x_l\rangle\langle x_l| \qquad (l = j-p) \\
&= \sum_{l=-\infty}^{\infty} la\,|x_l\rangle\langle x_l| + \sum_{l=-\infty}^{\infty} pa\,|x_l\rangle\langle x_l| \\
&= \sum_{l=-\infty}^{\infty} x_l\,|x_l\rangle\langle x_l| + pa\sum_{l=-\infty}^{\infty} |x_l\rangle\langle x_l| \\
&= \hat{X} + pa\hat{\mathbb{1}}
\end{aligned}
$$

3.7)

$$
\begin{aligned}
\hat{H}_0 &= -\frac{\hbar^2}{2ma^2}\left(\hat{T}_1 + \hat{T}_1^\dagger - 2\hat{\mathbb{1}}\right) \\
&= -\frac{\hbar^2}{2ma^2}\left(\hat{T}_1 + \hat{T}_1^{-1} - 2\hat{\mathbb{1}}\right) \\
&= -\frac{\hbar^2}{2ma^2}\left[\exp\left(\frac{i\hat{P}a}{\hbar}\right) + \exp\left(-\frac{i\hat{P}a}{\hbar}\right) - 2\hat{\mathbb{1}}\right] \\
&= -\frac{\hbar^2}{2ma^2}\left[\sum_{k=0}^{\infty}\frac{\left(\frac{ia}{\hbar}\right)^k}{k!}\hat{P}^k + \sum_{k=0}^{\infty}\frac{\left(-\frac{ia}{\hbar}\right)^k}{k!}\hat{P}^k - 2\hat{\mathbb{1}}\right] \\
&= -\frac{\hbar^2}{2ma^2}\left[\sum_{k=0}^{\infty}\frac{\left(\frac{a}{\hbar}\right)^k}{k!}\left(i^k + (-i)^k\right)\hat{P}^k - 2\hat{\mathbb{1}}\right] \\
&= -\frac{\hbar^2}{2ma^2}\left[(1+1)\,\hat{\mathbb{1}} + \frac{a}{\hbar}\,(i-i)\,\hat{P} + \frac{a^2}{2\hbar^2}\,(-1-1)\,\hat{P}^2 + \mathcal{O}\big(a^3\big) - 2\hat{\mathbb{1}}\right] \\
&= -\frac{\hbar^2}{2ma^2}\left[2\hat{\mathbb{1}} + 0 - \frac{a^2}{\hbar^2}\hat{P}^2 + \mathcal{O}\big(a^3\big) - 2\hat{\mathbb{1}}\right] \\
&= \frac{\hat{P}^2}{2m} + \mathcal{O}(a)
\end{aligned}
$$

$$
\Longrightarrow \lim_{a\to 0}\hat{H}_0 = \frac{\hat{P}^2}{2m}
$$

3.8)

$$\begin{aligned}
\hat{T}_1 |\theta\rangle &= \left(\sum_{j=-\infty}^{\infty} |x_{j-p}\rangle \langle x_j|\right)\left(\sum_{k=-\infty}^{\infty} e^{i\theta k} |x_k\rangle\right) \\
&= \sum_{j,k=-\infty}^{\infty} e^{i\theta k} |x_{j-1}\rangle \langle x_j \mid x_k\rangle \\
&= \sum_{j,k=-\infty}^{\infty} e^{i\theta k} |x_{j-1}\rangle \delta_{jk} \\
&= \sum_{j=-\infty}^{\infty} e^{i\theta j} |x_{j-1}\rangle \\
&= \sum_{l=-\infty}^{\infty} e^{i\theta (l+1)} |x_l\rangle && (l = j-1) \\
&= e^{i\theta} \sum_{l=-\infty}^{\infty} e^{i\theta l} |x_l\rangle \\
&= e^{i\theta} |\theta\rangle
\end{aligned}$$

Thus $|\theta\rangle$ are eigenstates of $\hat{T}_1$ with eigenvalues $e^{i\theta}$, $\theta \in [-\pi, \pi]$.

$$\begin{aligned}
\hat{T}_1^\dagger \left(\hat{T}_1 |\theta\rangle\right) &= \hat{T}_1^\dagger e^{i\theta} |\theta\rangle = e^{i\theta} \hat{T}_1^\dagger |\theta\rangle \\
\text{also} \quad \left(\hat{T}_1^\dagger \hat{T}_1\right) |\theta\rangle &= \hat{\mathbb{1}} |\theta\rangle \\
\implies \hat{T}_1^\dagger |\theta\rangle &= e^{-i\theta} |\theta\rangle
\end{aligned}$$

Thus $|\theta\rangle$ are also eigenstates of $\hat{T}_1^\dagger$ with eigenvalues $e^{-i\theta}$.

$$\begin{aligned}
\hat{H}_0 |\theta\rangle &= -\frac{\hbar^2}{2ma^2}\left(\hat{T}_1 |\theta\rangle + \hat{T}_1^\dagger |\theta\rangle - 2\hat{\mathbb{1}} |\theta\rangle\right) \\
&= -\frac{\hbar^2}{2ma^2}\left(e^{i\theta} + e^{-i\theta} - 2\right) |\theta\rangle
\end{aligned}$$

Thus $|\theta\rangle$ are eigenstates of $\hat{H}_0$ with eigenvalues $-\frac{\hbar^2}{2ma^2}\left(e^{i\theta} + e^{-i\theta} - 2\right) = \frac{\hbar^2}{ma^2}(1 - \cos\theta) = \frac{2\hbar^2}{ma^2} \sin^2 \frac{\theta}{2}$.

3.9)

$$\begin{aligned}
\hat{H}_1 &= -\frac{\hbar^2}{2ma^2}\left[\hat{T}_1 + \hat{T}_1^\dagger + \alpha\left(\hat{T}_1^2 - 2\hat{T}_1 + \hat{\mathbb{1}}\right) + \alpha^*\left(\hat{T}_1^{\dagger 2} - 2\hat{T}_1^\dagger + \hat{\mathbb{1}}\right) - 2\hat{\mathbb{1}}\right] \\
&= -\frac{\hbar^2}{2ma^2}\left[(\alpha + \alpha^* - 2)\,\hat{\mathbb{1}} + (1 - 2\alpha)\,\hat{T}_1 + (1 - 2\alpha^*)\,\hat{T}_1^\dagger + \alpha \hat{T}_1^2 + \alpha^* \hat{T}_1^{\dagger 2}\right] \\
\implies \hat{H}_1 |\theta\rangle &= -\frac{\hbar^2}{2ma^2}\left[(\alpha + \alpha^* - 2) + (1 - 2\alpha)\,e^{i\theta} + (1 - 2\alpha^*)\,e^{-i\theta} + \alpha \hat{T}_1 e^{i\theta} + \alpha^* \hat{T}_1^\dagger e^{-i\theta}\right] |\theta\rangle \\
&= -\frac{\hbar^2}{2ma^2}\left[(\alpha + \alpha^* - 2) + (1 - 2\alpha)\,e^{i\theta} + (1 - 2\alpha^*)\,e^{-i\theta} + \alpha e^{2i\theta} + \alpha^* e^{-2i\theta}\right] |\theta\rangle \\
&= -\frac{\hbar^2}{2ma^2}\left[\left(e^{i\theta} + e^{-i\theta} - 2\right) + \alpha\left(e^{2i\theta} - 2e^{i\theta} + 1\right) + \alpha^*\left(e^{-2i\theta} - 2e^{-i\theta} + 1\right)\right] |\theta\rangle \\
&= -\frac{\hbar^2}{2ma^2}\left[\left(e^{i\theta} + e^{-i\theta} - 2\right) + \alpha e^{i\theta}\left(e^{i\theta} + e^{-i\theta} - 2\right) + \alpha^* e^{-i\theta}\left(e^{i\theta} + e^{-i\theta} - 2\right)\right] |\theta\rangle \\
&= -\frac{\hbar^2}{2ma^2}\left(\alpha e^{i\theta} + \alpha^* e^{-i\theta} + 1\right)\left(e^{i\theta} + e^{-i\theta} - 2\right) |\theta\rangle
\end{aligned}$$

Thus $|\theta\rangle$ is an eigenstate of $\hat{H}_1$ with eigenvalues $-\frac{\hbar^2}{2ma^2}\left(\alpha e^{i\theta} + \alpha^* e^{-i\theta} + 1\right)\left(e^{i\theta} + e^{-i\theta} - 2\right)$.