MAU34601 Practical Numerical Simulations
Assignment 3 due 25/11/2022

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SS Theoretical Physics

1 Laplace equation with Dirichlet boundary conditions

A good choice for the over-relaxation parameter ω was found by taking a range of grid sizes and calculating the corresponding value of ω that resulted in the fastest convergence of the system. In particular, for a grid size $10m \times 10m$ and writing $\phi_\omega^{(k)}(x,y)$ as the value of the field at coordinate (x,y) at iteration k with over-relaxation paramter ω, the expressions

$$\Phi_\omega^{(k)} \equiv \sum_{(x,y)} \phi_\omega^{(k)}(x,y), \qquad k^* \equiv 15m^2,$$
$$\omega^* : \left| \Phi_{\omega^*}^{(k^*)} - \Phi_{\omega^*}^{(k^*-1)} \right| = \min_\omega \left| \Phi_\omega^{(k^*)} - \Phi_\omega^{(k^*-1)} \right|$$

were defined, i.e. the optimal over-relaxation parameter ω^* was chosen based on the difference in the sums of the field between successive iterations, dependent on the grid size.[1] By calculating and plotting ω^* for a range of grid sizes (Figure 3), a good choice of the over-relaxation parameter was found to be $\omega = 1.95$ for sufficiently large grid sizes.

The derivative $\frac{\partial \phi}{\partial y}(x,y)$ at a point can be written as

$$\phi(x,y+h) = \phi(x,y) + h\frac{\partial \phi}{\partial y}(x,y) + \frac{h^2}{2}\frac{\partial^2 \phi}{\partial y^2}(x,y) + \mathcal{O}(h^3)$$
$$\phi(x,y-h) = \phi(x,y) - h\frac{\partial \phi}{\partial y}(x,y) + \frac{h^2}{2}\frac{\partial^2 \phi}{\partial y^2}(x,y) + \mathcal{O}(h^3)$$
$$\implies \frac{\partial \phi}{\partial y}(x,y) = \frac{\phi(x,y+h) - \phi(x,y-h)}{2h} + \mathcal{O}(h^2),$$

leading to the $\mathcal{O}(h^2)$-accurate approximation

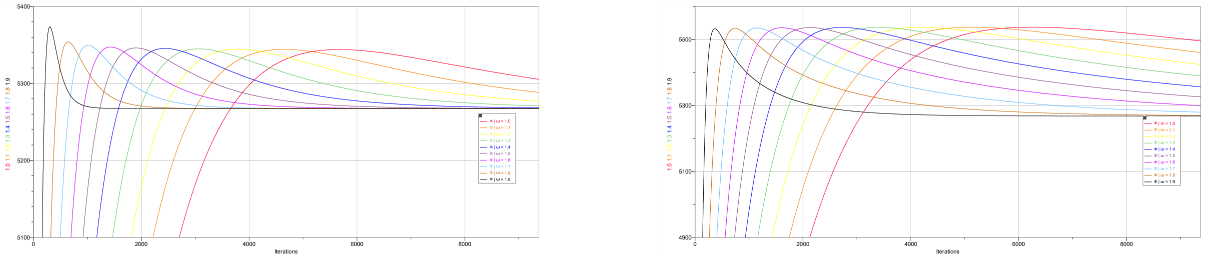
$$\frac{\partial \phi}{\partial y}\left(\frac{3}{10}, \frac{1}{2}\right) \approx \frac{\phi\left(\frac{3}{10}, \frac{1}{2} + h\right) - \phi\left(\frac{3}{10}, \frac{1}{2} - h\right)}{2h}.$$

Choosing a 1000×1000 grid size, $\omega = 1.95$, and $k = 20,000$, this was calculated to be $\frac{\partial \phi}{\partial y}\left(\frac{3}{10}, \frac{1}{2}\right) = 2.40$, to three significant figures.

2 Laplace equation with Neumann boundary conditions

The above was repeated for the case of Neumann boundary conditions on the left and lower boundaries, resulting in a good choice of the over-relaxation paramater of $\omega = 1.98$ and $\frac{\partial \phi}{\partial y}\left(\frac{3}{10}, \frac{1}{2}\right) = 2.49$, to three significant figures.

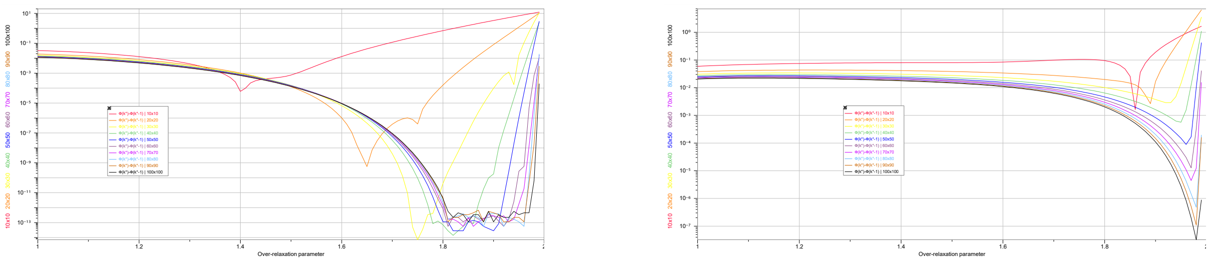
[1] This rather complicated set of expressions were defined as it was noticed that larger grid sizes required more iterations to reach a similar level of convergence. For each tested grid size it was found that after $15m^2$ iterations, suitable convergence had been reached for certain values of ω but not for others.



(a) Dirichlet boundary conditions.

(b) Neumann boundary conditions.

Figure 1: $\Phi_\omega^{(k)}$ against k for $\omega = 1.0, 1.01, \ldots, 1.99$, with 250×250 grid size.



(a) Dirichlet boundary conditions.

(b) Neumann boundary conditions.

Figure 2: $\left|\Phi_\omega^{(k^*)} - \Phi_\omega^{(k^*-1)}\right|$ against ω for grid sizes $100 \times 10, 20 \times 20, \ldots, 10 \times 100$.

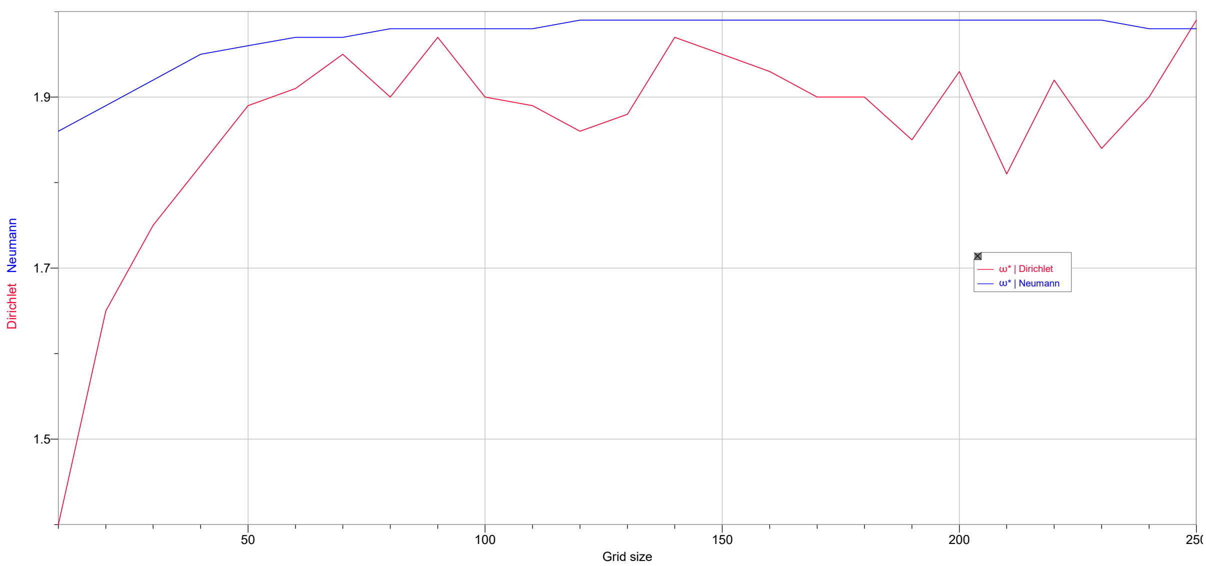


Figure 3: ω^* against grid sizes for Dirichlet and Neumann boundary conditions.

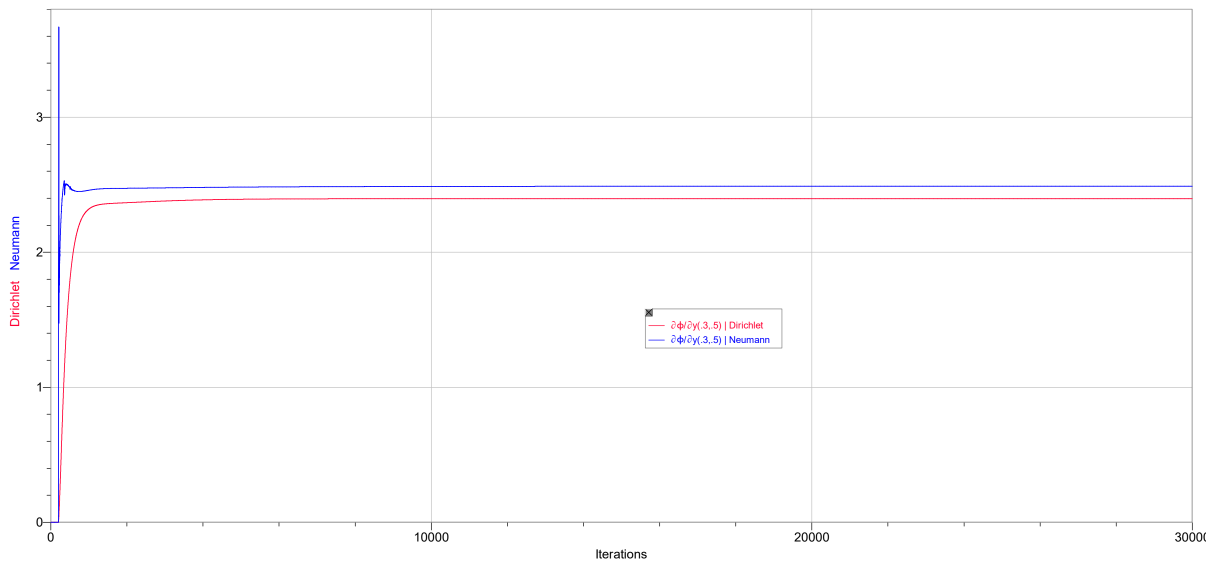


Figure 4: $\frac{\partial \phi}{\partial y}\left(\frac{3}{10}, \frac{1}{2}\right)$ against k for Dirichlet and Neumann boundary conditions, with 1000×1000 grid size and $\omega = 1.95, 1.98$.