MAU34601 Practical Numerical Simulations
Assignment 1 due 07/10/2022

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SS Theoretical Physics

1 Fourth Order Runge-Kutta

$$\frac{dx}{dt} = (1-t)(5-2t)\sqrt{1-x}$$

$$\int_{x_0}^{x} \frac{dx'}{\sqrt{1-x'}} = \int_{t_0}^{t} \left(5 - 7t' + 2(t')^2\right) dt'$$

$$-\int_{1-x_0}^{1-x} u^{-\frac{1}{2}} du = 5t' - \frac{7(t')^2}{2} + \frac{2(t')^3}{3}\Bigg|_{t_0}^{t} \qquad (v \equiv 1 - x')$$

$$-2u^{\frac{1}{2}}\Big|_{1-x_0}^{1-x} = \frac{2t^3}{3} - \frac{7t^2}{2} + 5t \qquad (t_0 = 0)$$

$$-2\sqrt{1-x} + 2\sqrt{1+\frac{3}{4}} = \frac{2t^3}{3} - \frac{7t^2}{2} + 5t \qquad (x_0 = -\tfrac{3}{4})$$

$$\sqrt{1-x} = -\frac{t^3}{3} + \frac{7t^2}{4} - \frac{5t}{2} + \sqrt{\frac{7}{4}}$$

$$x(t) = 1 - \left(\frac{t^3}{3} - \frac{7t^2}{4} + \frac{5t}{2} - \frac{\sqrt{7}}{2}\right)^2$$

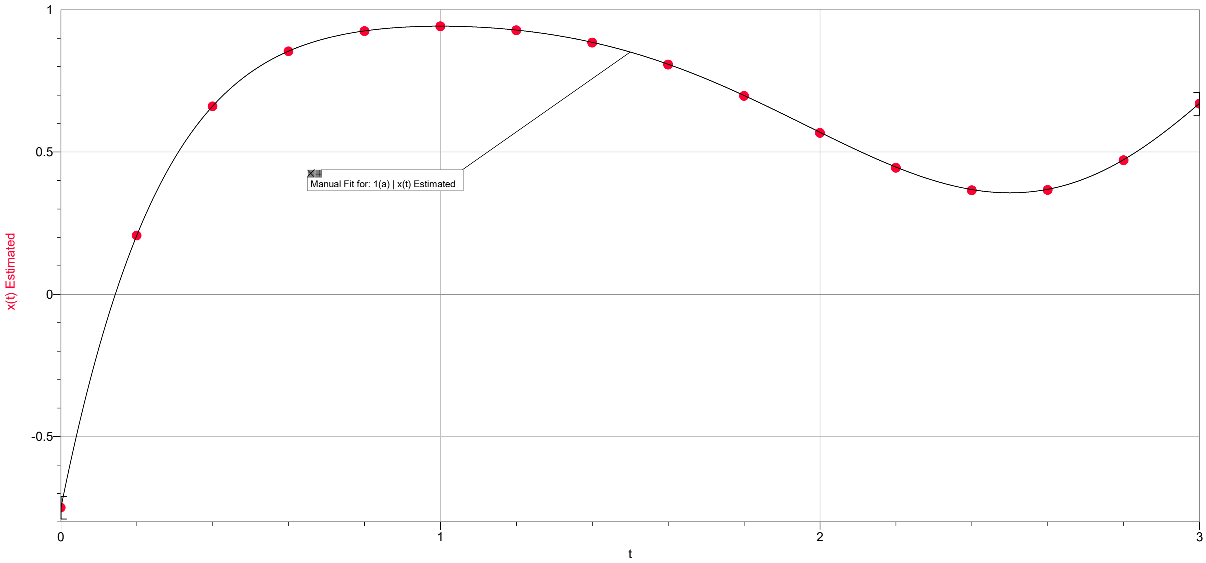


Figure 1: Plot of the approximate (red) and analytic (black) solutions of $x(t)$ for $t \in [0, 3]$, $h = 0.2$.

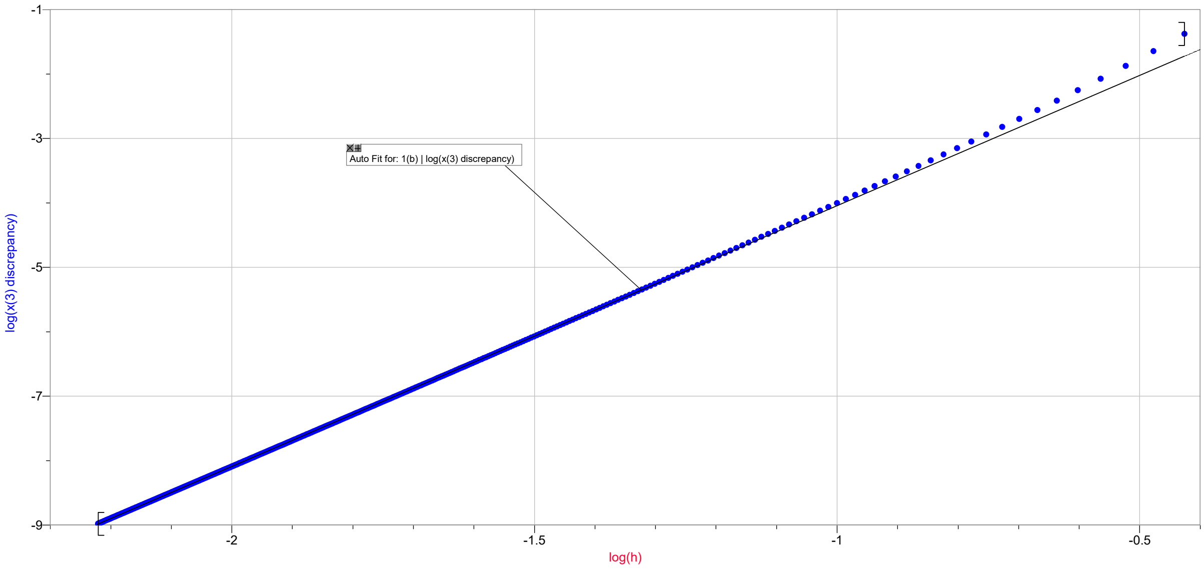


Figure 2: Plot of the logarithm of the discrepancy between estimated and analytic solution of $x(3)$ against the logarithm of the step-size h. The black line indicates a proportional line of slope 4.045.

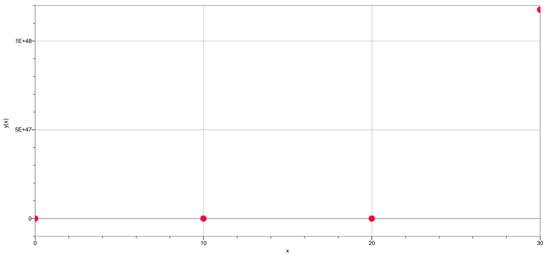
From Figure 2, it is clear that the fourth-order Runge-Kutta method has a fourth-order accuracy, as $\log|x_N - x(3)| \approx 4\log(h) \implies |x_N - x(3)| \approx h^4$.

N	h	x_N
8	0.375	0.62996119
9	0.33333333	0.64920548
10	0.3	0.65850788
⋮	⋮	⋮
297	0.01010101	0.67181347
298	0.010067114	0.67181348
299	0.010033445	0.67181348
⋮	⋮	⋮
500	0.006	0.67181348

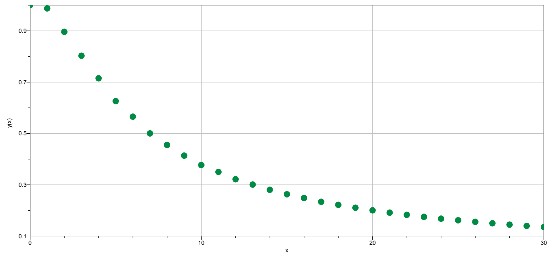
Table 1: Values of x_N of the Runge-Kutta method and their corresponding numbers of steps N and step-sizes h, to eight significant figures.

The analytic solution to the differential equation at $t = 3$ to eight significant figures is $x(t) = 0.67181348$. From Table 1, the fourth-order Runge-Kutta method reaches this value at $N = 298$, i.e. a step-size of $h = \frac{3}{298} = 0.010067114$ is required for a solution accurate to eight significant figures.

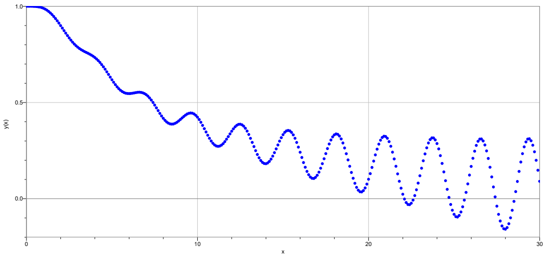
2 Runge-Kutta for higher-order ODEs



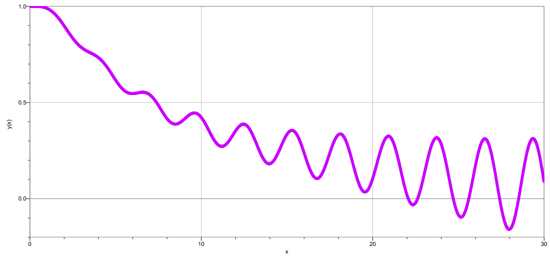
(a) $N = 3$, $h = 10$.



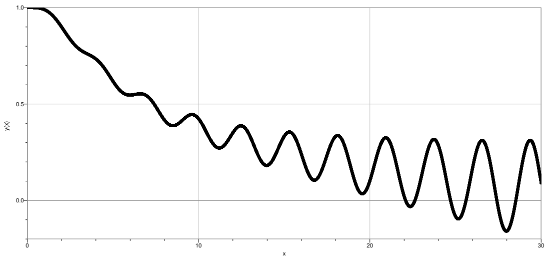
(b) $N = 30$, $h = 1$.



(c) $N = 300$, $h = 0.1$.



(d) $N = 3000$, $h = 0.01$.



(e) $N = 30000$, $h = 0.001$.

Figure 3: Plots of $y(x)$ for $x \in [0, 30]$ using the Runge-Kutta method for varying numbers of steps N and step-sizes h.

N	h	y_N
3	10	1.1759×10^{48}
30	1	0.135003
300	0.1	0.0905267
3000	0.01	0.0901524
30000	0.001	0.0901524

Table 2: Values of y_N of the Runge-Kutta method and their corresponding numbers of steps N and step-sizes h, to six significant figures.

From Table 2, for both $N = 3000$, $h = 0.01$ and $N = 30000$, $h = 0.001$, the Runge-Kutte algorithm approximates $y(30) = 0.0901524$, to six significant figures.