MAU44404 General Relativity
Homework 8 due 06/04/2023

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SS Theoretical Physics

Problem 4*

$$\text{Fixed time } t = T \implies dt = 0$$
$$\text{Radial path } \implies d\theta = d\phi = 0$$

$$\begin{aligned}
ds_1^2 &= T^2 \frac{dr^2}{\left(1+r^2\right)^2} \\
ds_1 &= T \frac{dr}{1+r^2} \\
l_1 &= T \int_0^R \frac{dr}{1+r^2} \\
&= T\left(\arctan R - \arctan 0\right) \\
l_1 &= T \arctan R \\
&\in \left[-\frac{\pi T}{2}, \frac{\pi T}{2}\right]
\end{aligned}$$

$$\begin{aligned}
ds_2^2 &= T^{\frac{4}{3}} \frac{dr^2}{\left(1-r^2\right)^2} \\
ds_2 &= T^{\frac{2}{3}} \frac{dr}{1-r^2} \\
l_2 &= T^{\frac{2}{3}} \int_0^R \frac{dr}{1-r^2} \\
&= T^{\frac{2}{3}}\left(\operatorname{arctanh} R - \operatorname{arctanh} 0\right) \\
l_2 &= T^{\frac{2}{3}} \operatorname{arctanh} R \\
&\in (-\infty, \infty)
\end{aligned}$$

$$\begin{aligned}
ds_3^2 &= T^2 \frac{dr^2}{1-r^2} \\
ds_3^2 &= T \frac{dr}{\sqrt{1-r^2}} \\
l_3 &= T \int_0^R \frac{dr}{\sqrt{1-r^2}} \\
&= T\left(\arcsin R - \arcsin 0\right) \\
l_3 &= T \arcsin R \\
&\in \left[-\frac{\pi T}{2}, \frac{\pi T}{2}\right]
\end{aligned}$$

As ds_1 and ds_3 correspond to bounded proper distances for a fixed time T, they could represent a closed (finite) univserse. As ds_2 corresponds to an unbounded proper distance, it could represent an open (infinite) universe.

Problem 5*

$$
\begin{aligned}
0 &= D_\mu(\rho U^\mu) + p D_\mu U^\mu \\
&= U^\mu D_\mu \rho + (\rho + p)\, D_\mu U^\mu \\
&= U^\mu \partial_\mu \rho + \frac{\rho + p}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, U^\mu\right) \\
&= \partial_t \rho + \frac{\rho + p}{\sqrt{-g}} \partial_t \sqrt{-g} + 0
\end{aligned}
$$

$$
\begin{aligned}
ds^2 &= -dt^2 + a^2 \, \frac{dx^2 + dy^2 + dz^2}{\left(1 + \frac{kr^2}{4}\right)^2} \\
\implies g &= -1 \left(\frac{a^2}{\left(1 + \frac{kr^2}{4}\right)^2} \right)^3 \\
\implies \sqrt{-g} &= \left(\frac{a}{1 + \frac{kr^2}{4}} \right)^3 \\
\implies \partial_t \sqrt{-g} &= 3 \left(\frac{a}{1 + \frac{kr^2}{4}} \right)^2 \frac{\dot{a}}{1 + \frac{kr^2}{4}} \\
&= 3 \sqrt{-g}\, \frac{\dot{a}}{a}
\end{aligned}
$$

$$
\begin{aligned}
\implies 0 &= \dot{\rho} + \frac{\rho + p}{\sqrt{-g}}\, 3\sqrt{-g}\, \frac{\dot{a}}{a} \\
&= \dot{\rho} + 3 \frac{\dot{a}}{a} \, (\rho + p)
\end{aligned}
$$