MAU44404 General Relativity
Homework 7 due 31/03/2023

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SS Theoretical Physics

Problem 4*

$$x_1 = a\sin\theta_1 \qquad \implies dx_1 = a\cos\theta_1\,d\theta_1$$
$$x_j = a\alpha_j\cos\theta_j,\ j = 2,3,4,5 \qquad \implies dx_j = a\left(\cos\theta_1\,d\alpha_j - \alpha_j\sin\theta_1\,d\theta_1\right)$$

$$\begin{aligned}
a^2 &= \sum_{i=1}^{5} x_i^2 \\
&= a^2\sin^2\theta_1 + \sum_{j=2}^{5} a^2\alpha_j^2\cos^2\theta_1 \\
\implies 1 &= \sin^2\theta_1 + \cos^2\theta_1\sum_{j=2}^{5}\alpha_j^2 \\
\implies \sum_{j=2}^{5}\alpha_j^2 &= 1 \\
\implies \sum_{j=2}^{5}\alpha_j\,d\alpha_j &= 0
\end{aligned}$$

$$\begin{aligned}
ds^2 &= \sum_{i=1}^{5} dx_i^2 \\
&= a^2\cos^2\theta_1\,d\theta_1^2 + \sum_{j=2}^{5} a^2\left(\cos^2\theta_1\,d\alpha_j^2 + \alpha_j^2\sin^2\theta_1\,d\theta_1^2 - 2\alpha_j\cos\theta_1\sin\theta_1\,d\alpha_j\,d\theta_1\right) \\
&= a^2\left[\left(\cos^2\theta_1 + \sin^2\theta_1\sum_{j=2}^{5}\alpha_j^2\right)d\theta_1^2 + \cos^2\theta_1\sum_{j=2}^{5}d\alpha_j^2 - 2\cos\theta_1\sin\theta_1\,d\theta_1\sum_{j=2}^{5}\alpha_j\,d\alpha_j\right] \\
&= a^2\left(d\theta_1^2 + \cos^2\theta_1\sum_{j=2}^{5}d\alpha_j^2 - 0\right)
\end{aligned}$$

$$\alpha_2 = \cos\theta_2 \qquad \implies d\alpha_2 = -\sin\theta_2\,d\theta_2$$
$$\alpha_k = \beta_k\sin\theta_2,\ k = 3,4,5 \qquad \implies d\alpha_k = \sin\theta_2\,d\beta_k + \beta_k\cos\theta_2\,d\theta_2$$

$$
\begin{aligned}
1 &= \sum_{j=2}^{5} \alpha_j^2 \\
&= \cos^2\theta_2 + \sin^2\theta_2 \sum_{k=3}^{5} \beta_k^2 \\
\implies \sum_{k=3}^{5} \beta_k^2 &= 1 \\
\implies \sum_{k=3}^{5} \beta_k \, d\beta_k &= 0
\end{aligned}
$$

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\begin{aligned}
\sum_{j=2}^{5} d\alpha_j^2 &= \sin^2\theta_2\, d\theta_2^2 + \sum_{k=3}^{5} \left(\sin^2\theta_2\, d\beta_k^2 + \beta_k^2 \cos^2\theta_2\, d\theta_2^2 + 2\beta_k \sin\theta_2 \cos\theta_2\, d\beta_k\, d\theta_2\right) \\
&= \left(\sin^2\theta_2 + \cos^2\theta_2 \sum_{k=3}^{5} \beta_k^2\right) d\theta_2^2 + \sin^2\theta_2^2 \sum_{k=3}^{5} d\beta_k^2 + 2\sin\theta_2\cos\theta_2\, d\theta_2 \sum_{k=3}^{5} \beta_k\, d\beta_k \\
&= d\theta_2^2 + \sin^2\theta_2 \sum_{k=3}^{5} d\beta_k^2 + 0 \\
\implies ds^2 &= a^2 \left[d\theta_1^2 + \cos\theta_1^2 \left(d\theta_2^2 + \sin^2\theta_2 \sum_{k=3}^{5} d\beta_k^2\right)\right]
\end{aligned}
$$

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\begin{aligned}
\beta_3 &= \sin\theta_3 \cos\theta_4 && \implies d\beta_3 = \cos\theta_3\cos\theta_4\, d\theta_3 - \sin\theta_3 \sin\theta_4\, d\theta_4 \\
\beta_4 &= \sin\theta_3 \sin\theta_4 && \implies d\beta_4 = \cos\theta_3\sin\theta_4\, d\theta_3 + \sin\theta_3 \cos\theta_4\, d\theta_4 \\
\beta_5 &= \cos\theta_3 && \implies d\beta_5 = -\sin\theta_3\, d\theta_3
\end{aligned}
$$

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\begin{aligned}
\sum_{k=3}^{5} \beta_k^2 &= \sin^2\theta_3\cos^2\theta_4 + \sin^2\theta_3\sin^2\theta_4 + \cos^2\theta_3 \\
&= \sin^2\theta_3 \left(\cos^2\theta_4 + \sin^2\theta_4\right) + \cos^2\theta_3 \\
&= 1 \implies \text{coordinate changes meets the constraint}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=3}^{5} d\beta_k^2 =&\ \cos^2\theta_3\cos^2\theta_4\, d\theta_3^2 + \sin^2\theta_3\sin^2\theta_4\, d\theta_4^2 - 2\cos\theta_3\cos\theta_4\sin\theta_3\sin\theta_4\, d\theta_3\, d\theta_4 \\
&+ \cos^2\theta_3\sin^2\theta_4\, d\theta_3^2 + \sin^2\theta_3\cos^2\theta_4\, d\theta_4^2 + 2\cos\theta_3\sin\theta_4\sin\theta_3\cos\theta_4\, d\theta_3\, d\theta_4 \\
&+ \sin^2\theta_3\, d\theta_3^2 \\
=&\ \left[\cos^2\theta_3\left(\cos^2\theta_4 + \sin^2\theta_4\right) + \sin^2\theta_3\right] d\theta_3^2 + \sin^2\theta_3\left(\cos^2\theta_4 + \sin^2\theta_4\right) d\theta_4^2 + 0 \\
=&\ d\theta_3^2 + \sin^2\theta_3\, d\theta_4^2 \\
\implies ds^2 =&\ a^2 \left\{d\theta_1^2 + \cos^2\theta_1 \left[d\theta_2^2 + \sin^2\theta_2 \left(d\theta_3^2 + \sin^2\theta_3\, d\theta_4^2\right)\right]\right\}
\end{aligned}
$$

Problem 5*

$$x_5 = ix_0 \qquad \Longrightarrow ds^2 = -dx_0^2 + \sum_{i=1}^{4} dx_i^2, \qquad a^2 = -x_0^2 + \sum_{i=1}^{4} x_i^2$$

$$\begin{aligned} x_0 &= \sqrt{a^2 - r^2}\sinh t && \Longrightarrow dx_0 = \sqrt{a^2 - r^2}\cosh t\, dt - \frac{r}{\sqrt{a^2 - r^2}}\sinh t\, dr \\ x_1 &= \sqrt{a^2 - r^2}\cosh t && \Longrightarrow dx_1 = \sqrt{a^2 - r^2}\sinh t\, dt - \frac{r}{\sqrt{a^2 - r^2}}\cosh t\, dr \end{aligned}$$

$$\begin{aligned} -dx_0^2 + dx_1^2 = &- \left(a^2 + r^2\right)\cosh^2 t\, dt^2 - \frac{r^2}{a^2 - r^2}\sinh^2 t\, dr^2 + 2r\sinh t\cosh t\, dt\, dr \\ &+ \left(a^2 + r^2\right)\sinh^2 t\, dt^2 + \frac{r^2}{a^2 - r^2}\cosh^2 t\, dr^2 - 2r\sinh t\cosh t\, dt\, dr \\ = &- \left(a^2 - r^2\right) dt^2 + \frac{r^2}{a^2 - r^2}\, dr^2 \end{aligned} \tag{1}$$

$$\begin{aligned} -x_0^2 + x_1^2 &= -\left(a^2 - r^2\right)\sinh^2 t + \left(a^2 - r^2\right)\cosh^2 t \\ &= a^2 - r^2 \\ \Longrightarrow a^2 &= a^2 - r^2 + \sum_{j=2}^{4} x_j^2 \\ \Longrightarrow r^2 &= \sum_{j=2}^{4} x_j^2 \end{aligned}$$

$$\begin{aligned} x_2 &= r\sin\theta\cos\phi && \Longrightarrow dx_2 = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi \\ x_3 &= r\sin\theta\sin\phi && \Longrightarrow dx_3 = \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi \\ x_4 &= r\cos\theta && \Longrightarrow dx_4 = \cos\theta\, dr - r\sin\theta\, d\theta \end{aligned}$$

$$\begin{aligned} \sum_{j=2}^{4} x_j^2 &= r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta \\ &= r^2\left[\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\right] \\ &= r^2 \implies \text{coordinate change meets the constraint} \end{aligned}$$

$$\begin{aligned} \sum_{i=2}^{4} dx_i^2 = &\ \sin^2\theta\cos^2\phi\, dr^2 + r^2\cos^2\theta\cos^2\phi\, d\theta^2 + r^2\sin^2\theta\sin^2\phi\, d\phi^2 \\ &+ 2r\sin\theta\cos\theta\cos^2\phi\, dr\, d\theta - 2r\sin^2\theta\cos\phi\sin\phi\, dr\, d\phi - 2r^2\cos\theta\sin\theta\cos\phi\sin\phi\, d\theta\, d\phi \\ &+ \sin^2\theta\sin^2\phi\, dr^2 + r^2\cos^2\theta\sin^2\phi\, d\theta^2 + r^2\sin^2\theta\cos^2\phi\, d\phi^2 \\ &+ 2r\sin\theta\cos\theta\sin^2\phi\, dr\, d\theta + 2r\sin^2\theta\sin\phi\cos\phi\, dr\, d\phi + 2r^2\cos\theta\sin\theta\sin\phi\cos\phi\, d\theta\, d\phi \\ &+ \cos^2\theta\, dr^2 + r^2\sin^2\theta\, d\theta^2 - 2r\cos\theta\sin\theta\, dr\, d\theta \\ = &\ \left[\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\right] dr^2 + r^2\left[\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \sin^2\theta\right] d\theta^2 \\ &+ r^2\sin^2\theta\left(\sin^2\phi + \cos^2\phi\right) d\phi^2 + 2r\sin\theta\cos\theta\left(\cos^2\phi + \sin^2\phi - 1\right) dr\, d\theta + 0 + 0 \\ = &\ dr^2 + r^2\, d\theta^2 + r^2\sin^2\theta\, d\phi^2 \end{aligned} \tag{2}$$

$$\begin{aligned} (1),(2) \implies ds^2 &= -\left(a^2 - r^2\right) dt^2 + \frac{r^2}{a^2 - r^2}\, dr^2 + dr^2 + r^2\, d\theta^2 + r^2\sin^2\theta\, d\phi^2 \\ &= -\left(a^2 - r^2\right) dt^2 + \frac{a^2}{a^2 - r^2}\, dr^2 + r^2 d\Omega_2^2 \end{aligned}$$