MAU44404 General Relativity
Homework 6 due 23/03/2023

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SS Theoretical Physics

Problem 4*

$$\begin{aligned} n_\mu &\equiv \partial_\mu S \\ \implies n_t = n_x &= 1 \\ n_\mu n^\mu &= g^{\mu\nu} n_\mu n_\nu \\ &= -1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 \\ &= 0 \end{aligned}$$

Thus the normal vectors of S are null. As the tangent vectors of null surfaces are also normal vectors, then the tangent vectors of S are null, and so S is a null surface.

Problem 5*

(a)

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r}\right) dt^2 && \text{(for fixed } r, \theta, \phi) \\ -1 &= \left(\frac{ds}{d\tau}\right)^2 && \text{(for a timelike interval)} \\ &= -\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 \\ \implies \frac{d\tau}{dt} &= \left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}} \end{aligned}$$

$$\implies \frac{d\tau_A}{dt} = \left(1 - \frac{2GM}{r_A}\right)^{\frac{1}{2}}, \qquad \frac{d\tau_B}{dt} = \left(1 - \frac{2GM}{r_B}\right)^{\frac{1}{2}}$$

(b)

$$\begin{aligned}
\omega_A \lambda_A &= \omega_B \lambda_B \\
\implies \frac{\omega_A}{\omega_B} &= \frac{\lambda_B}{\lambda_A} \\
&= \frac{T_B}{T_A} \\
&= \frac{d\tau_B}{d\tau_A} \\
&= \frac{d\tau_B}{dt}\frac{dt}{d\tau_A} \\
&= \left(\frac{1 - \frac{2GM}{r_B}}{1 - \frac{2GM}{rB}} \right)^{\frac{1}{2}} \\
\frac{\omega_A}{\omega_B} &= \left(\frac{r_A}{r_B} \frac{r_B - 2GM}{r_A - 2GM} \right)^{\frac{1}{2}} \\
z &\equiv \frac{\lambda_B}{\lambda_A} - 1 \\
&= \frac{\omega_A}{\omega_B} - 1 \\
z &= \left(\frac{r_A}{r_B} \frac{r_B - 2GM}{r_A - 2GM} \right)^{\frac{1}{2}} - 1
\end{aligned}$$

(c)

As $r_A \to 2GM$ for fixed $r_B > 2GM$, we have that $z \to \infty$. This corresponds to an observer outside the event horizon detecting an infinite redshift from a source at the event horizon.