MAU44404 General Relativity
Homework 5 due 16/03/2023

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SS Theoretical Physics

Problem 4*

(a)

$$\begin{aligned} V(r) &= \frac{1}{2}\left(\varepsilon + \frac{l^2}{r^2}\right)\left(1 - \frac{2GM}{r}\right) \\ &= \frac{\varepsilon}{2} - \frac{GM\varepsilon}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} \\ \implies \frac{dV(r)}{dr} &= \frac{GM\varepsilon}{r^2} - \frac{l^2}{r^3} + \frac{3GMl^2}{r^4} \end{aligned} \tag{1}$$

$$\begin{aligned} \text{Null geodesics} &\implies \varepsilon = 0 \\ \text{Circular orbit} &\implies \frac{dV(r)}{dr} = 0 \\ (1) &\implies -\frac{l^2}{r^3} + \frac{3GMl^2}{r^4} = 0 \\ &\implies r = 3GM \end{aligned}$$

(b)

$$\begin{aligned} \text{Timelike geodesics} &\implies \varepsilon = 1 \\ \text{Circular orbit} &\implies \frac{dV(r)}{dr} = 0 \\ (1) &\implies \frac{GM}{r^2} - \frac{l^2}{r^3} + \frac{3GMl^2}{r^4} = 0 \\ &\implies GMr^2 - l^2 r + 3GMl^2 = 0 \\ &\implies r_\pm = \frac{l^2 \pm \sqrt{l^4 - 12G^2M^2l^2}}{2GM} \end{aligned}$$

$$\begin{aligned} \text{No orbit} &\implies r_\pm \text{ does not exist} \\ &\implies l^4 - 12G^2M^2l^2 < 0 \\ &\implies l < 2\sqrt{3}\,GM \end{aligned}$$

$$\begin{aligned} \text{One orbit} &\implies r_+ = r_- \\ &\implies l^4 - 12G^2M^2l^2 = 0 \\ &\implies l = 2\sqrt{3}\,GM \end{aligned}$$

(c)

$$
\begin{aligned}
r_{\pm} &= \frac{l^2 \pm \sqrt{l^4 - 12G^2M^2l^2}}{2GM} \\
&= \frac{l^2}{2GM}\left(1 \pm \sqrt{1 - \frac{12G^2M^2}{l^2}}\right) \\
&= \frac{l^2}{2GM}\left[1 \pm \left(1 - \frac{6G^2M^2}{l^2} + \mathcal{O}\left(l^{-4}\right)\right)\right] \\
&\approx \frac{l^2}{2GM}(1 \pm 1) \mp 3GM
\end{aligned}
$$

$$
\frac{d^2V(r)}{dr^2} = -\frac{2GM}{r^3} + \frac{3l^2}{r^4} - \frac{12GMl^2}{r^5}
$$

$$
\begin{aligned}
\left.\frac{d^2V(r)}{dr^2}\right|_{r_+} &\approx -\frac{2GM}{\left(\frac{l^2}{GM} - 3GM\right)^3} + \frac{3l^2}{\left(\frac{l^2}{GM} - 3GM\right)^4} - \frac{12GMl^2}{\left(\frac{l^2}{GM} - 3GM\right)^5} \\
&= \frac{3l^2 - 2GM\left(\frac{l^2}{GM} - 3GM\right)}{\left(\frac{l^2}{GM} - 3GM\right)^4} - \frac{12GMl^2}{\left(\frac{l^2}{GM} - 3GM\right)^5} \\
&= \frac{l^2}{\left(\frac{l^2}{GM} - 3GM\right)^4} + \mathcal{O}(l^{-8}) \\
&\gtrsim 0 \implies r_+ \text{ corresponds to a stable orbit}
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{d^2V(r)}{dr^2}\right|_{r_-} &\approx -\frac{2}{27G^2M^2} + \frac{l^2}{27G^4M^4} - \frac{4l^2}{81G^4M^4} \\
&= -\left(\frac{l^2}{81G^4M^4} + \frac{2}{27G^2M^2}\right) \\
&< 0 \implies r_- \text{ corresponds to an unstable orbit}
\end{aligned}
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Problem 5*

(a)

$$
\begin{aligned}
\text{Radial geodesics} &\implies d\theta = d\phi = 0 \\
\text{Null geodesics} &\implies ds^2 = 0 \\
&\implies -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = 0 \\
&\implies dt^2 = \left(1 - \frac{2GM}{r}\right)^{-2} dr^2 \\
&\implies \frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}
\end{aligned}
$$

(b)

$$\begin{aligned}
\frac{dt}{dr} &= \pm\left(1-\frac{2GM}{r}\right)^{-1}\\
&= \pm\left(\frac{r}{r-2GM}\right)\\
\implies t_{\pm} &= \pm\int dr\left(\frac{r}{r-2GM}\right)\\
&= \pm\int dr'\left(\frac{r'+2GM}{r'}\right) \qquad\qquad (r' = r-2GM)\\
&= \pm\left(r'+2GM\ln r'\right)+\text{const.}\\
&= \pm\left(r-2GM+2GM\ln(r-2GM)\right)+\text{const.}\\
&= \pm\left(r+2GM\ln\Big(\frac{r}{2GM}-1\Big)+2GM\ln(2GM)\right)+\text{const.}\\
&= \pm\left(r+2GM\ln\Big(\frac{r}{2GM}-1\Big)\right)+\text{const.}
\end{aligned}$$

$$t_+ - r - 2GM\ln\Big(\frac{r}{2GM}-1\Big) = \text{const.} \equiv v_0 \qquad\qquad t_- + r + 2GM\ln\Big(\frac{r}{2GM}-1\Big) = \text{const.} \equiv u_0$$

Thus the null geodesics are given by $u = u_0 = \text{const.}$ and $v = v_0 = \text{const.}$

(c)

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)$$

$$\begin{aligned}
du &= dt+dr+2GM\left(\frac{r}{2GM}-1\right)^{-1}\frac{dr}{2GM}\\
&= dt+\left(1+\frac{2GM}{r-2GM}\right)dr\\
&= dt+\frac{r}{r-2GM}\,dr\\
&= dt+\left(1-\frac{2GM}{r}\right)^{-1}dr\\
\text{Similarly, } dv &= dt-\left(1-\frac{2GM}{r}\right)^{-1}dr
\end{aligned}$$

$$\begin{aligned}
&\implies du\,dv = dt^2-\left(1-\frac{2GM}{r}\right)^{-2}dr^2\\
&\implies -\left(1-\frac{2GM}{r}\right)dt^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2 = -\left(1-\frac{2GM}{r}\right)du\,dv
\end{aligned}$$

$$\implies ds^2 = -\left(1-\frac{2GM}{r}\right)du\,dv+r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)$$

$$U = e^{\frac{u}{4GM}} \implies dU = e^{\frac{u}{4GM}} \, \frac{du}{4GM} \qquad\qquad V = -e^{-\frac{v}{4GM}} \implies dV = -e^{-\frac{v}{4GM}} \, \frac{(-dv)}{4GM}$$

$$\implies du = \frac{4GM}{U} \, du \qquad\qquad \implies dv = -\frac{4GM}{V} \, dV$$

$$\implies du \, dv = -(4GM)^2 \, \frac{dU \, dV}{UV}$$

$$\implies ds^2 = \left(1 - \frac{2GM}{r}\right) (4GM)^2 \, \frac{dU \, dV}{UV} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$$\begin{aligned} UV &= -e^{\frac{u-v}{4GM}} \\ &= -\exp\left[\frac{1}{4GM} \left(2r + 4GM \ln\left(\frac{r}{2GM} - 1\right)\right)\right] \\ &= -\left(\frac{r}{2GM} - 1\right) e^{\frac{r}{2GM}} \end{aligned}$$

$$\begin{aligned} \implies ds^2 &= -\left(1 - \frac{2GM}{r}\right) (4GM)^2 \left(\frac{r}{2GM} - 1\right)^{-1} e^{-\frac{r}{2GM}} \, dU \, dV + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \\ &= -16G^2M^2 \, \frac{r - 2GM}{r} \, \frac{2GM}{r - 2GM} \, e^{-\frac{r}{2GM}} \, dU \, dV + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \\ &= -\frac{32G^3M^3}{r} \, e^{-\frac{r}{2GM}} \, dU \, dV + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \end{aligned}$$

$$U = T + X \implies dU = dT + dX \qquad\qquad V = T - X \implies dV = dT - dX$$

$$\implies -dU \, dV = -dT^2 + dX^2$$

$$\implies ds^2 = \frac{32G^3M^3}{r} \, e^{-\frac{r}{2GM}} \left(-dT^2 + dX^2\right) + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$