MAU44404 General Relativity
Homework 4 due 28/02/2023

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Problem 3*

$$\begin{aligned}\frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}V^\mu\big) &= \frac{\sqrt{-g}}{\sqrt{-g}}\partial_\mu V^\mu + \frac{1}{\sqrt{-g}}V^\mu\partial_\mu\big(\sqrt{-g}\big) \\ &= \partial_\mu V^\mu + \frac{1}{2}V^\nu\left(\frac{1}{g}\partial_\nu g\right)\end{aligned}$$

$$\begin{aligned}\det e^A = e^{\operatorname{tr} A} &\implies \det(g_{\mu\lambda}) = e^{\operatorname{tr}\ln(g_{\mu\lambda})} \\ &\implies \ln g = \operatorname{tr}\ln(g_{\mu\lambda}) \\ &\implies \partial_\nu \ln g = \partial_\nu \operatorname{tr}\ln(g_{\mu\lambda}) \\ &\implies \frac{1}{g}\partial_\nu g = \operatorname{tr}\partial_\nu \ln(g_{\mu\lambda}) \\ &\qquad\qquad = \operatorname{tr}\big(g^{\mu\lambda}\partial_\nu g_{\lambda\sigma}\big) \\ &\qquad\qquad = g^{\mu\lambda}\partial_\nu g_{\mu\lambda}\end{aligned}$$

$$\implies \frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}V^\mu\big) = \partial_\mu V^\mu + \frac{1}{2}V^\nu g^{\mu\lambda}\partial_\nu g_{\mu\lambda}$$

$$\begin{aligned}D_\mu V^\mu &= \partial_\mu V^\mu + \Gamma^\mu_{\mu\nu}V^\nu \\ &= \partial_\mu V^\mu + \frac{1}{2}g^{\mu\lambda}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)V^\nu \\ &= \partial_\mu V^\mu + \frac{1}{2}\left(\partial^\lambda g_{\lambda\nu} - \partial^\mu g_{\mu\nu} + g^{\mu\lambda}\partial_\nu g_{\mu\lambda}\right)V^\nu \\ &= \partial_\mu V^\mu + \frac{1}{2}V^\nu g^{\mu\lambda}\partial_\nu g_{\mu\lambda} \\ &= \frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}V^\mu\big)\end{aligned}$$

Problem 4*

(a)

$$\begin{aligned}
T_{\mu\nu} &= -2\frac{\delta\left(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right)}{\delta g^{\mu\nu}} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\
&= \frac{1}{2}\frac{\delta\left(g^{\alpha\rho}g^{\beta\sigma}\right)}{\delta g^{\mu\nu}}F_{\alpha\beta}F_{\rho\sigma} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\
&= \frac{1}{2}\left(\frac{\delta g^{\alpha\rho}}{\delta g^{\mu\nu}}g^{\beta\sigma} + g^{\alpha\rho}\frac{\delta g^{\beta\sigma}}{\delta g^{\mu\nu}}\right)F_{\alpha\beta}F_{\rho\sigma} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\
&= \frac{1}{2}\left(g^{\beta\sigma}F_{\mu\beta}F_{\nu\sigma} + g^{\alpha\rho}F_{\alpha\mu}F_{\rho\nu}\right) - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\
&= \frac{1}{2}\left(g^{\beta\sigma}F_{\mu\beta}F_{\nu\sigma} + g^{\alpha\rho}F_{\mu\alpha}F_{\nu\rho}\right) - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\
&= F_{\mu}{}^{\sigma}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}
\end{aligned}$$

(b)

$$\begin{aligned}
\delta S &= \int d^4x\,\delta\left[\sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\right] \\
&= -\frac{1}{4}\int d^4x\,\sqrt{-g}\,\delta(F_{\mu\nu}F_{\rho\sigma})\,g^{\mu\rho}g^{\nu\sigma} \\
&= -\frac{1}{4}\int d^4x\,\sqrt{-g}\,[(\delta F_{\mu\nu})\,F_{\rho\sigma} + F_{\mu\nu}\,(\delta F_{\rho\sigma})]\,g^{\mu\rho}g^{\nu\sigma} \\
&= -\frac{1}{2}\int d^4x\,\sqrt{-g}\,F^{\mu\nu}\delta F_{\mu\nu} \\
&= -\frac{1}{2}\int d^4x\,\sqrt{-g}\,F^{\mu\nu}\left(\delta\partial_\mu A_\nu - \delta\partial_\nu A_\mu\right) \\
&= -\int d^4x\,\sqrt{-g}F^{\mu\nu}\partial_\mu\delta A_\nu \\
&= \int d^4x\,\partial_\mu\left(\sqrt{-g}\,F^{\mu\nu}\right)\delta A_\nu && \text{(integration by parts)} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\implies 0 &= \partial_\mu\left(\sqrt{-g}\,F^{\mu\nu}\right) \\
&= \sqrt{-g}\left[\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,F^{\mu\nu}\right)\right] \\
&= \sqrt{-g}\,D_\mu F^{\mu\nu} && \text{(Problem 3*)} \\
\implies D_\mu F^{\mu\nu} &= 0
\end{aligned}$$

(c)

$$
\begin{aligned}
T_{\mu\nu} &= g^{\beta\sigma} F_{\mu\beta} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} g^{\alpha\rho} g^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma} \\
D^{\mu} T_{\mu\nu} &= g^{\mu\lambda} D_{\lambda} T_{\mu\nu} \\
&= g^{\mu\lambda} \left[g^{\beta\sigma} D_{\lambda} (F_{\mu\beta} F_{\nu\sigma}) - \frac{1}{4} g_{\mu\nu} g^{\alpha\rho} g^{\beta\sigma} D_{\lambda} (F_{\alpha\beta} F_{\rho\sigma}) \right] \\
&= g^{\mu\lambda} g^{\beta\sigma} (D_{\lambda} F_{\mu\beta}) F_{\nu\sigma} + g^{\mu\lambda} g^{\beta\sigma} F_{\mu\beta} (D_{\lambda} F_{\nu\sigma}) - \frac{1}{4} \delta^{\lambda}_{\nu} g^{\alpha\rho} g^{\beta\sigma} \left[(D_{\lambda} F_{\alpha\beta}) F_{\rho\sigma} + F_{\alpha\beta} (D_{\lambda} F_{\rho\sigma}) \right] \\
&= F_{\nu}{}^{\beta} D^{\mu} F_{\mu\beta} + F^{\lambda\sigma} D_{\lambda} F_{\nu\sigma} - \frac{1}{2} g^{\alpha\rho} g^{\beta\sigma} F_{\rho\sigma} D_{\nu} F_{\alpha\beta} \\
&= 0 + F^{\lambda\sigma} D_{\lambda} F_{\nu\sigma} - \frac{1}{2} F^{\alpha\beta} D_{\nu} F_{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
D_{[\lambda} F_{\nu\sigma]} = 0 &\implies D_{\lambda}(F_{\nu\sigma} - F_{\sigma\nu}) + D_{\nu}(F_{\sigma\lambda} - F_{\lambda\sigma}) + D_{\sigma}(F_{\lambda\nu} - F_{\nu\lambda}) = 0 \\
&\implies 2D_{\lambda} F_{\nu\sigma} - 2D_{\nu} F_{\lambda} \sigma - 2D_{\sigma} F_{\nu\lambda} = 0 \\
&\implies F^{\lambda\sigma} D_{\lambda} F_{\nu\sigma} - F^{\lambda\sigma} D_{\nu} F_{\lambda\sigma} - F^{\lambda\sigma} D_{\sigma} F_{\nu\lambda} = 0 \\
&\implies F^{\lambda\sigma} D_{\lambda} F_{\nu\sigma} - F^{\lambda\sigma} D_{\nu} F_{\lambda\sigma} + F^{\sigma\lambda} D_{\sigma} F_{\nu\lambda} = 0 \\
&\implies 2F^{\lambda\sigma} D_{\lambda} F_{\nu\sigma} - F^{\lambda\sigma} D_{\nu} F_{\lambda\sigma} = 0 \\
&\implies F^{\lambda\sigma} D_{\lambda} F_{\nu\sigma} = \frac{1}{2} F^{\lambda\sigma} D_{\nu} F_{\lambda\sigma}
\end{aligned}
$$

$$
\begin{aligned}
\implies D^{\mu} T_{\mu\nu} &= \frac{1}{2} F^{\lambda\sigma} D_{\nu} F_{\lambda\sigma} - \frac{1}{2} F^{\alpha\beta} D_{\nu} F_{\alpha\beta} \\
&= 0
\end{aligned}
$$