MAU44404 General Relativity
Homework 3 due 21/02/2023

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SS Theoretical Physics

Problem 3*

(a)

$$\begin{aligned}
&D_{(\mu}\epsilon_{\nu)} = 0 \\
\implies &\partial_{(\mu}\epsilon_{\nu)} - \Gamma^{\lambda}_{(\mu\nu)}\epsilon_{\lambda} = 0 && (\Gamma^{\alpha}_{\beta\gamma} = 0) \\
\implies &\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = 0 \\
\implies &\partial_{\rho}\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\partial_{\rho}\epsilon_{\mu} = 0 && (1) \\
\implies &\partial_{\mu}\partial_{\nu}\epsilon_{\rho} + \partial_{\rho}\partial_{\mu}\epsilon_{\nu} = 0 && (2) \\
\implies &\partial_{\nu}\partial_{\rho}\epsilon_{\mu} + \partial_{\mu}\partial_{\nu}\epsilon_{\rho} = 0 && (3)
\end{aligned}$$

$$\begin{aligned}
(1) - (2) - (3) \implies &-2\partial_{\mu}\partial_{\nu}\epsilon_{\rho} = 0 \\
\implies &\partial_{\mu}\partial_{\nu}\epsilon_{\rho} = 0
\end{aligned}$$

(b)

$$\begin{aligned}
&\partial_{\rho}\partial_{\mu}\epsilon_{\nu} = 0 \\
\implies &\partial_{\mu}\epsilon_{\nu} = M_{\mu\nu} && (\text{with } M_{\mu\nu} \text{ independent of } x) \\
\implies &\epsilon_{\nu} = M_{\mu\nu}x^{\mu} + P_{\nu} && (\text{with } P_{\nu} \text{ independent of } x)
\end{aligned}$$

Since we have $D_{(\mu}\epsilon_{\nu)} = \partial_{(\mu}\epsilon_{\nu)} = 0$ and $\partial_{\mu}\epsilon_{\nu} = M_{\mu\nu}$ we have $M_{(\mu\nu)} = 0$. Thus $M_{\mu\nu}$ is independent of x and antisymmetric, and P_{ν} is independent of x.

(c)

P_{ν} corresponds to translations and $M_{\mu\nu}$ to rotations and Lorentz boosts. For example, translations in the x-direction are given by

$$(P_{\nu}) = \begin{pmatrix} \varepsilon \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

boosts in the x-direction are given by

$$(M_{\mu\nu}) = \begin{pmatrix} -\gamma & \gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and rotations about the x-axis are given by

$$(M_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

(d)

P_ν has four independent components (translation in each direction), and $M_{\mu\nu}$ has 6 independent components (rotation and boost in each spatial direction). Thus there are ten linearly independent Killing vectors. From Problem 1, a 4-dimensional space can have a maximum of $\frac{4(4+1)}{2} = 10$ independent Killing vectors, and so Minkowski space is maximally symmetric.

Problem 4*

(a)

From Homework 2 Problem 3,

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \implies \hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\nu \partial_\mu \ln|\Omega| + \delta^\lambda_\mu \partial_\nu \ln|\Omega| - g_{\mu\nu}\partial^\lambda \ln|\Omega|$$

Thus in the context of this question, with

$$\begin{aligned} \Omega \to e^\Phi &\implies \partial_\alpha \ln|\Omega| \to \partial_\alpha \Phi, \\ g_{\mu\nu} \to \eta_{\mu\nu} &\implies \Gamma^\lambda_{\mu\nu} \to {}^{(\eta)}\Gamma^\lambda_{\mu\nu} = 0, \\ \hat{g}_{\mu\nu} \to g_{\mu\nu} &\implies \hat{\Gamma}^\lambda_{\mu\nu} \to {}^{(g)}\Gamma^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu}, \end{aligned}$$

we have

$$\Gamma^\lambda_{\mu\nu} = \delta^\lambda_\nu \partial_\mu \Phi + \delta^\lambda_\mu \partial_\nu \Phi - \eta_{\mu\nu}\partial^\lambda \Phi.$$

$$\begin{aligned} R &\equiv g^{\mu\nu} R_{\mu\lambda\nu}{}^\lambda \\ &= e^{-2\Phi}\eta^{\mu\nu}\left(\partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\lambda\nu} + \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\lambda\sigma} - \Gamma^\sigma_{\lambda\nu}\Gamma^\lambda_{\mu\sigma}\right) \end{aligned} \tag{4}$$

$$\begin{aligned} \partial_\lambda \Gamma^\lambda_{\mu\nu} &= \partial_\lambda\left(\delta^\lambda_\nu \partial_\mu \Phi + \delta^\lambda_\mu \partial_\nu \Phi - \eta_{\mu\nu}\partial^\lambda \Phi\right) \\ &= \partial_\nu \partial_\mu \Phi + \partial_\mu \partial_\nu \Phi - \eta_{\mu\nu}\partial_\lambda \partial^\lambda \Phi \\ \implies \eta^{\mu\nu}\partial_\lambda \Gamma^\lambda_{\mu\nu} &= \partial_\nu \partial^\nu \Phi + \partial_\mu \partial^\mu \Phi - 4\partial_\lambda \partial^\lambda \Phi \\ &= -2\partial_\mu \partial^\mu \Phi \end{aligned} \tag{5}$$

$$\begin{aligned} \partial_\mu \Gamma^\lambda_{\lambda\nu} &= \partial_\mu(\partial_\nu \Phi + 4\partial_\nu \Phi - \partial_\nu \Phi) \\ &= 4\partial_\mu \partial_\nu \Phi \\ \implies \eta^{\mu\nu}\partial_\mu \Gamma^\lambda_{\lambda\nu} &= 4\partial_\mu \partial^\mu \Phi \end{aligned} \tag{6}$$

$$\begin{aligned} \Gamma^\sigma_{\mu\nu} &= \delta^\sigma_\nu \partial_\mu \Phi + \delta^\sigma_\mu \partial_\nu \Phi - \eta_{\mu\nu}\partial^\sigma \Phi \\ \implies \eta^{\mu\nu}\Gamma^\sigma_{\mu\nu} &= \partial^\sigma \Phi + \partial^\sigma \Phi - 4\partial^\sigma \Phi \\ &= -2\partial^\sigma \Phi \end{aligned} \qquad \begin{aligned} \Gamma^\lambda_{\lambda\sigma} &= \partial_\sigma \Phi + 4\partial_\sigma \Phi - \partial_\sigma \Phi \\ &= 4\partial_\sigma \Phi \end{aligned}$$

$$\begin{aligned} \implies \eta^{\mu\nu}\Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\lambda\sigma} &= \left(-2\partial^\sigma\right)\left(4\partial_\sigma \Phi\right) \\ &= -8\partial_\sigma \Phi \partial^\sigma \Phi \end{aligned} \tag{7}$$

$$\Gamma^{\sigma}_{\lambda\nu} = \delta^{\sigma}_{\nu}\partial_{\lambda}\Phi + \delta^{\sigma}_{\lambda}\partial_{\nu}\Phi - \eta_{\mu\nu}\partial^{\sigma}\Phi \qquad \Gamma^{\lambda}_{\mu\sigma} = \delta^{\lambda}_{\sigma}\partial_{\mu}\Phi + \delta^{\lambda}_{\mu}\partial_{\sigma}\Phi - \eta_{\mu\sigma}\partial^{\lambda}\Phi$$
$$\implies \eta^{\mu\nu}\Gamma^{\sigma}_{\lambda\nu} = \eta^{\mu\sigma}\partial_{\lambda}\Phi + \delta^{\sigma}_{\lambda}\partial^{\mu}\Phi - \delta^{\mu}_{\lambda}\partial^{\sigma}$$

$$\begin{aligned}\implies \eta^{\mu\nu}\Gamma^{\sigma}_{\lambda\nu}\Gamma^{\lambda}_{\mu\sigma} =& \left(\eta^{\mu\sigma}\partial_{\lambda}\Phi + \delta^{\sigma}_{\lambda}\partial^{\mu}\Phi - \delta^{\mu}_{\lambda}\partial^{\sigma}\right)\left(\delta^{\lambda}_{\sigma}\partial_{\mu}\Phi + \delta^{\lambda}_{\mu}\partial_{\sigma}\Phi - \eta_{\mu\sigma}\partial^{\lambda}\Phi\right)\\ =& \,\partial_{\lambda}\Phi\partial^{\lambda}\Phi + \partial_{\lambda}\Phi\partial^{\lambda}\Phi - 4\partial_{\lambda}\Phi\partial^{\lambda}\Phi\\ &+ 4\partial_{\mu}\Phi\partial^{\mu}\Phi + \partial_{\mu}\Phi\partial^{\mu}\Phi - \partial_{\mu}\Phi\partial^{\mu}\Phi\\ &- \partial_{\mu}\Phi\partial^{\mu}\Phi - 4\partial_{\sigma}\Phi\partial^{\sigma}\Phi + \partial_{\mu}\Phi\partial^{\mu}\Phi\\ =& -2\partial_{\mu}\Phi\partial^{\mu}\Phi \end{aligned} \tag{8}$$

$$\begin{aligned}(4),(5),(6),(7),(8) \implies R &= e^{-2\Phi}\left(-2\partial_{\mu}\partial^{\mu}\Phi - 4\partial_{\mu}\partial^{\mu}\Phi - 8\partial_{\sigma}\Phi\partial^{\sigma}\Phi + 2\partial_{\mu}\Phi\partial^{\mu}\Phi\right)\\ &= -6e^{-2\Phi}\left(\partial_{\mu}\partial^{\mu}\Phi + \partial_{\mu}\Phi\partial^{\mu}\phi\right)\end{aligned}$$

(b)

$$\begin{aligned} g_{\mu\nu} &= e^{2\Phi}\eta_{\mu\nu}\\ &\approx \eta_{\mu\nu} + 2\Phi\eta_{\mu\nu}\\ &= \eta_{\mu\nu} + h_{\mu\nu}\\ \implies h_{\mu\nu} &= 2\Phi\eta_{\mu\nu}\end{aligned}$$

$$\begin{aligned}\Gamma^{\mu}_{\nu\rho} &= \frac{1}{2}\eta^{\mu\sigma}\left(\partial_{\nu}h_{\sigma\rho} + \partial_{\rho}h_{\sigma\nu} - \partial_{\sigma}h_{\nu\rho}\right)\\ &= \eta^{\mu\sigma}\left(\eta_{\sigma\rho}\partial_{\nu}\Phi + \eta_{\sigma\nu}\partial_{\rho}\Phi - \eta_{\nu\rho}\partial_{\sigma}\Phi\right)\\ \Gamma^{0}_{00} &= 0\\ \Gamma^{i}_{00} &= \eta^{ii}\left(\eta_{i0}\partial_{0}\Phi + \eta_{i0}\partial_{0}\Phi - \eta_{00}\partial_{i}\Phi\right)\\ &= \partial_{i}\Phi\end{aligned}$$

Thus, similarly to the lecture notes, we have $\frac{d^2x^i}{dt^2} = -\partial_i\Phi$, and so the Φ in this question is the gravitational potential. For a slow moving particle we have

$$T_{\mu\nu} = \begin{cases}\rho, & \mu=\nu=0\\ 0 & \text{otherwise}\end{cases}.$$

$$\begin{aligned}R &= \kappa^2 g^{00}T_{00}\\ &= -e^{-2\Phi}\kappa^2\rho\end{aligned} \qquad \begin{aligned}R &= -6e^{-2\Phi}\left(\partial_{\mu}\partial^{\mu}\Phi + \partial_{\mu}\Phi\partial^{\mu}\Phi\right)\\ &\approx -6e^{-2\Phi}\partial_{\mu}\partial^{\mu}\Phi\end{aligned}$$

$$\begin{aligned}&\implies \Box\Phi = \frac{1}{6}\kappa^2\rho\\ &\implies \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \frac{1}{6}\kappa^2\rho\end{aligned}$$

For a spherically symmetric mass we have

$$\rho(r) = \begin{cases}\rho_0, & r \geqslant R\\ 0, & r < R\end{cases}.$$

$$\begin{aligned}r^2\frac{\partial\Phi^{(\text{out})}}{\partial r} &= A\\ \implies \Phi^{(\text{out})}(r) &= -\frac{A}{r} + B\end{aligned} \qquad \begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi^{(\text{in})}}{\partial r}\right) &= \frac{1}{6}\kappa^2\rho_0\\ \Phi^{(\text{in})}(r) &= \frac{1}{36}\kappa^2\rho_0 r^2\end{aligned}$$

We need $\Phi(r)$ and its derivative to be continuous at $r = R$.

$$\begin{aligned}
\left.\frac{\partial\Phi^{(\text{out})}}{\partial r}\right|_{r=R} &= \left.\frac{\partial\Phi^{(\text{in})}}{\partial r}\right|_{r=R} \\
\implies \frac{A}{R^2} &= \frac{1}{18}\kappa^2\rho_0 R \\
\implies A &= \frac{1}{18}\kappa^2\rho_0 R^3
\end{aligned}$$

$$\begin{aligned}
\implies \Phi^{(\text{out})}(r) &= -\frac{1}{18}\kappa^2\rho_0\frac{R^3}{r} + \text{const} \\
&= -\frac{GM}{r} + \text{const} \\
\implies \frac{1}{18}\kappa^2\frac{M}{\frac{4}{3}\pi R^3}R^3 &= GM \\
\implies \kappa^2 &= 24\pi G
\end{aligned}$$

(c)

$$\begin{aligned}
\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta &= 0 \\
\implies \ddot{x}^\mu + \left(\delta^\mu_\beta\partial\alpha\Phi + \delta^\mu_\alpha\partial_\beta\Phi - \eta_{\alpha\beta}\partial^\mu\Phi\right)\dot{x}^\alpha\dot{x}^\beta &= 0 \\
\implies \ddot{x}^\mu + \dot{x}^\mu\dot{x}^\alpha\partial_\alpha\Phi + \dot{x}^\mu\dot{x}^\beta\partial_\beta\Phi - \eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\partial^\mu\Phi &= 0 \\
\implies \ddot{x}^\mu + 2\left(\dot{x}^\nu\partial_\nu\Phi\right)\dot{x}^\mu &= 0 \qquad \text{(Null geodesics} \implies \text{last term} = 0) \\
\implies \ddot{x}^\mu &= -2\left(\dot{x}^\nu\partial_\nu\Phi\right)\dot{x}^\mu
\end{aligned}$$

As the equation for $\ddot{x}^\mu$ is linear in $\dot{x}^\mu$, then the corresponding path of the geodesic is a straight line, and so is not deflected by the presence of any mass.