MAU44404 General Relativity
Homework 2 due 13/02/2023

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SS Theoretical Physics

Problem 3*

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\begin{aligned}
\hat{\Gamma}^{\lambda}_{\mu\nu} &= \frac{1}{2}\hat{g}^{\lambda\rho}\left(\partial_\mu \hat{g}_{\rho\nu} + \partial_\nu \hat{g}_{\rho\mu} - \partial_\rho \hat{g}_{\mu\nu}\right) \\
&= \frac{1}{2\Omega^2} g^{\lambda\rho}\left[\partial_\mu\left(\Omega^2 g_{\rho\nu}\right) + \partial_\nu\left(\Omega^2 g_{\mu\rho}\right) - \partial_\rho\left(\Omega^2 g_{\mu\nu}\right)\right] \qquad \left(\hat{g}_{\alpha\beta}\hat{g}^{\beta\gamma} = \delta^\gamma_\alpha \implies \hat{g}^{\alpha\beta} = \tfrac{1}{\Omega^2} g^{\alpha\beta}\right) \\
&= \frac{1}{2\Omega^2} g^{\lambda\rho}\left(g_{\rho\nu}\partial_\mu\Omega^2 + \Omega^2\partial_\mu g_{\rho\nu} + g_{\mu\rho}\partial_\nu\Omega^2 + \Omega^2\partial_\nu g_{\mu\rho} - g_{\mu\nu}\partial_\rho\Omega^2 - \Omega^2\partial_\rho g_{\mu\nu}\right) \\
&= \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) + \frac{1}{2\Omega^2} g^{\lambda\rho}\left(2\Omega g_{\rho\nu}\partial_\mu\Omega + 2\Omega g_{\mu\rho}\partial_\nu\Omega - 2\Omega g_{\mu\nu}\partial_\rho\Omega\right) \\
&= \Gamma^\lambda_{\mu\nu} + \frac{1}{\Omega}\left(\delta^\lambda_\nu\partial_\mu\Omega + \delta^\lambda_\mu\partial_\nu\Omega - g_{\mu\nu}\partial^\lambda\Omega\right) \\
&= \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\nu\partial_\mu \ln|\Omega| + \delta^\lambda_\mu\partial_\nu \ln|\Omega| - g_{\mu\nu}\partial^\lambda \ln|\Omega|
\end{aligned}
$$

Problem 5*

$$
\begin{aligned}
D^\mu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) &= g^{\mu\lambda}D_\lambda\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) \\
&= g^{\mu\lambda}D_\lambda R_{\mu\nu} - \frac{1}{2}D_\nu R \qquad \left(g^{\mu\lambda}D_\lambda g_{\mu\nu}R = D_\lambda g^{\mu\lambda}g_{\mu\nu}R = D_\nu R\right)
\end{aligned}
$$

$$
\begin{aligned}
D_\nu R &= D_\nu g^{\mu\lambda}R_{\mu\lambda} \\
&= g^{\mu\lambda}D_\nu R_{\mu\rho\lambda}{}^\rho \\
&= g^{\mu\lambda}\left(-D_\mu R_{\rho\nu\lambda}{}^\rho - D_\rho R_{\nu\mu\lambda}{}^\rho\right) \qquad \left(D_{[\nu}R_{\mu\rho]\lambda}{}^\rho = 0 \implies D_\nu R_{\mu\rho\lambda}{}^\rho = -D_\mu R_{\rho\nu\lambda}{}^\rho - D_\rho R_{\nu\mu\lambda}{}^\rho\right) \\
&= g^{\mu\lambda}\left(D_\mu R_{\nu\rho\lambda}{}^\rho - D_\rho R_{\nu\mu\lambda}{}^\rho\right) \qquad \left(R_{\rho\nu\lambda}{}^\rho = -R_{\nu\rho\lambda}{}^\rho\right) \\
&= g^{\mu\lambda}\left(D_\mu R_{\nu\rho\lambda}{}^\rho - g^{\sigma\rho}D_\rho R_{\nu\mu\lambda\sigma}\right) \\
&= g^{\mu\lambda}\left(D_\mu R_{\nu\rho\lambda}{}^\rho - g^{\sigma\rho}D_\rho R_{\lambda\sigma\nu\mu}\right) \qquad \left(R_{\nu\mu\lambda\sigma} = R_{\lambda\sigma\nu\mu}\right) \\
&= g^{\mu\lambda}D_\mu R_{\nu\rho\lambda}{}^\rho - g^{\sigma\rho}D_\rho R_{\lambda\sigma\nu}{}^\lambda \\
&= g^{\mu\lambda}D_\mu R_{\nu\rho\lambda}{}^\rho + g^{\sigma\rho}D_\rho R_{\sigma\lambda\nu}{}^\lambda \qquad \left(R_{\lambda\sigma\nu}{}^\lambda = -R_{\sigma\lambda\nu}{}^\lambda\right) \\
&= g^{\mu\lambda}D_\mu R_{\nu\lambda} + g^{\sigma\rho}D_\rho R_{\sigma\nu} \qquad \left(R_{\alpha\beta} \equiv R_{\alpha\gamma\beta}{}^\gamma\right) \\
&= g^{\lambda\mu}D_\mu R_{\lambda\nu} + g^{\sigma\rho}D_\rho R_{\sigma\nu} \qquad \left(g^{\mu\lambda} = g^{\lambda\mu},\ R_{\nu\lambda} = R_{\lambda\nu}\right) \\
&= g^{\mu\lambda}D_\lambda R_{\mu\nu} + g^{\mu\lambda}D_\lambda R_{\mu\nu} \qquad \text{(relabelling dummy indices)} \\
&= 2g^{\mu\lambda}D_\lambda R_{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
\implies D^\mu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) &= g^{\mu\lambda}D_\lambda R_{\mu\nu} - \frac{1}{2}\left(2g^{\mu\lambda}D_\lambda R_{\mu\nu}\right) \\
&= 0
\end{aligned}
$$

Problem 7*

$$ds^2 = 2\,du\,dv + H(x,y)\,du^2 + dx^2 + dy^2$$

$$\Longrightarrow\ (g_{\mu\nu}) = \begin{pmatrix} H(x,y) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -H(x,y) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
\Gamma^\nu_{\mu\lambda} &\equiv \frac{1}{2} g^{\nu\rho}\left(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}\right) \\
&= \frac{1}{2} g^{\nu\rho}\left[\delta^u_\rho \delta^u_\lambda \left(\delta^x_\mu \partial_x H + \delta^y_\mu \partial_y H\right) + \delta^u_\rho \delta^u_\mu \left(\delta^x_\lambda \partial_x H + \delta^y_\lambda \partial_y H\right) - \delta^u_\mu \delta^u_\lambda \left(\delta^x_\rho \partial_x H + \delta^y_\rho \partial_y H\right)\right] \\
&\qquad\qquad (\partial_\alpha g_{\beta\gamma} = \delta^u_\beta \delta^u_\gamma \left(\delta^x_\alpha \partial_x H + \delta^y_\alpha \partial_y H\right)) \\
&= \frac{1}{2}\left\{g^{\nu u}\left[\delta^u_\lambda \left(\delta^x_\mu \partial_x H + \delta^y_\mu \partial_y H\right) + \delta^u_\mu \left(\delta^x_\lambda \partial_x H + \delta^y_\lambda \partial_y H\right)\right] - \delta^u_\mu \delta^u_\lambda \left(g^{\nu x}\partial_x H + g^{\nu y}\partial_y H\right)\right\} \\
&= \frac{1}{2}\left\{\delta^\nu_v \left[\delta^u_\lambda \left(\delta^x_\mu \partial_x H + \delta^y_\mu \partial_y H\right) + \delta^u_\mu \left(\delta^x_\lambda \partial_x H + \delta^y_\lambda \partial_y H\right)\right] - \delta^u_\mu \delta^u_\lambda \left(\delta^\nu_x \partial_x H + \delta^\nu_y \partial_y H\right)\right\} \\
&\qquad\qquad (g^{\nu u} = \delta^\nu_v,\ g^{\nu x} = \delta^\nu_x,\ g^{\nu y} = \delta^\nu_y)
\end{aligned}$$

$$\begin{aligned}
R_{\mu\nu\lambda}{}^\alpha &\equiv \partial_\nu \Gamma^\alpha_{\mu\lambda} - \partial_\mu \Gamma^\alpha_{\nu\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\alpha_{\nu\beta} - \Gamma^\beta_{\nu\lambda}\Gamma^\alpha_{\mu\beta} \\
R_{\mu\lambda} &\equiv R_{\mu\nu\lambda}{}^\nu \\
&= \partial_\nu \Gamma^\nu_{\mu\lambda} - \partial_\mu \Gamma^\nu_{\nu\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\nu_{\nu\beta} - \Gamma^\beta_{\nu\lambda}\Gamma^\nu_{\mu\beta}
\end{aligned}$$

From inspection, the only non-vanishing $\Gamma^\nu_{\mu\lambda}$ are Γ^v_{xu}, Γ^v_{yu}, Γ^v_{ux}, Γ^v_{uy}, Γ^x_{uu}, Γ^y_{uu}. Thus $\Gamma^\nu_{\nu\lambda} = \Gamma^\nu_{\nu\beta} = 0$, and so we have

$$R_{\mu\lambda} = \partial_\nu \Gamma^\nu_{\mu\lambda} - 0 + 0 - \Gamma^\beta_{\nu\lambda}\Gamma^\nu_{\mu\beta}.$$

As the only non-vanishing partial derivatives of $\Gamma^\nu_{\mu\lambda}$ are those with respect to x or y, the sum over $\nu = u, v, x, y$ in the first term reduces to a sum over $\nu = x, y$, and so

$$R_{\mu\lambda} = \partial_x \Gamma^x_{\mu\lambda} + \partial_y \Gamma^y_{\mu\lambda} - \Gamma^\beta_{\nu\lambda}\Gamma^\nu_{\mu\beta}.$$

Since $\Gamma^u_{\rho\sigma} = \Gamma^\rho_{v\sigma} = 0$, the sums over $\beta, \nu = u, v, x, y$ in the last term reduce to sums over $\beta, \nu = x, y$. However, we also have that $\Gamma^x_{x\lambda} = \Gamma^x_{y\lambda} = \Gamma^y_{x\lambda} = \Gamma^y_{y\lambda} = 0$, and so the last term vanishes, resulting in

$$R_{\mu\lambda} = \partial_x \Gamma^x_{\mu\lambda} + \partial_y \Gamma^y_{\mu\lambda}.$$

As the only non-vanishing $\Gamma^\nu_{\mu\lambda}$ for $\nu = x, y$ are Γ^x_{uu} and Γ^y_{uu}, we have that $R_{\mu\lambda} = 0$ if $(\mu, \lambda) \neq (u, u)$.

$$\Gamma^x_{uu} = -\frac{1}{2}\partial_x H \qquad\qquad \Gamma^y_{uu} = -\frac{1}{2}\partial_y H$$

$$\Longrightarrow\ R_{uu} = -\frac{1}{2}\left(\partial_x^2 + \partial_y^2\right) H$$

If $H(x,y)$ satisfies $\left(\partial_x^2 + \partial_y^2\right) H(x,y) = 0$, then $R_{\mu\lambda} = 0$, and so $R \equiv g_{\mu\lambda} R^{\mu\lambda} = 0$, and thus Einstein's equations reduce to $T_{\mu\lambda} = 0$, i.e. the vacuum equations.