MAU44404 General Relativity
Homework 1 due 07/02/2023

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Problem 2*

(a)

$$
\begin{aligned}
{T'^{\mu\nu}}_{\lambda} &= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial x'^{\lambda}} {T^{\alpha\beta}}_{\gamma} && ({T^{\mu\nu}}_{\lambda} \text{ transforms as a } (2,\,1) \text{ tensor}) \\
\implies {T'^{\mu\nu}}_{\mu} &= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial x'^{\mu}} {T^{\alpha\beta}}_{\gamma} && (\lambda \to \mu) \\
\implies \widehat{T}'^{\nu} &= \left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\gamma}}{\partial x'^{\mu}} \right) \frac{\partial x'^{\nu}}{\partial x^{\beta}} {T^{\alpha\beta}}_{\gamma} && (\widehat{T}'^{\nu} \equiv {T'^{\mu\nu}}_{\mu}) \\
&= \delta^{\gamma}_{\alpha} \frac{\partial x'^{\nu}}{\partial x^{\beta}} {T^{\alpha\beta}}_{\gamma} && (\tfrac{\partial x^{\alpha}}{\partial y^{\gamma}} \tfrac{\partial y^{\gamma}}{\partial x^{\beta}} = \tfrac{\partial x^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta}) \\
&= \frac{\partial x'^{\nu}}{\partial x^{\beta}} {T^{\alpha\beta}}_{\alpha} && (\delta^{\gamma}_{\alpha} \neq 0 \Leftrightarrow \gamma = \alpha) \\
&= \frac{\partial x'^{\nu}}{\partial x^{\beta}} \widehat{T}^{\beta} && (\widehat{T}^{\beta} \equiv {T^{\alpha\beta}}_{\alpha})
\end{aligned}
$$

Thus $\widehat{T}^{\nu}$ transforms as a (1, 0) tensor.

(b)

$$
\begin{aligned}
F_{\rho\mu} A^{\rho} + F_{\mu\sigma} A^{\sigma} &= F_{\rho\mu} A^{\rho} - F_{\sigma\mu} A^{\sigma} && (F_{\mu\sigma} = -F_{\sigma\mu}) \\
&= F_{\sigma\mu} A^{\sigma} - F_{\sigma\mu} A^{\sigma} && (\rho \to \sigma) \\
&= 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
F_{\mu\nu} T^{\mu\nu} &= \frac{1}{2} \left(F_{\mu\nu} T^{\mu\nu} + F_{\mu\nu} T^{\mu\nu} \right) \\
&= \frac{1}{2} \left(F_{\mu\nu} T^{\mu\nu} - F_{\nu\mu} T^{\mu\nu} \right) && (F_{\mu\nu} = -F_{\nu\mu}) \\
&= \frac{1}{2} \left(F_{\mu\nu} T^{\mu\nu} - F_{\nu\mu} T^{\nu\mu} \right) && (T^{\mu\nu} = T^{\nu\mu}) \\
&= \frac{1}{2} \left(F_{\mu\nu} T^{\mu\nu} - F_{\mu\nu} T^{\mu\nu} \right) && (\mu \leftrightarrow \nu) \\
&= 0
\end{aligned}
$$

Problem 4*

$$
\begin{aligned}
D'_\nu V'^\mu &= \partial'_\nu V'^\mu + \Gamma'^\mu_{\nu\lambda} V'^\lambda \\
&= \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial}{\partial x^\beta}\left(\frac{\partial x'^\mu}{\partial x^\alpha} V^\alpha\right) + \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\Gamma^\alpha_{\beta\sigma} V'^\lambda - \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\alpha} V'^\lambda \\
&= \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\alpha} V^\alpha + \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial V^\alpha}{\partial x^\beta} + \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\nu}\Gamma^\alpha_{\beta\sigma}\left(\frac{\partial x^\sigma}{\partial x'^\lambda} V'^\lambda\right) - \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\alpha}\left(\frac{\partial x^\alpha}{\partial x'^\lambda} V'^\lambda\right) \\
&= \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\alpha} V^\alpha + \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\nu}\left(\partial_\beta V^\alpha + \Gamma^\alpha_{\beta\sigma} V^\sigma\right) - \frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\alpha} V^\alpha \\
&= \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\nu} D_\beta V^\alpha \implies \text{covariant derivatives of vectors are tensors}
\end{aligned}
$$

$$
\begin{aligned}
D'_\nu W'_\mu &= \partial'_\nu W'_\mu - \Gamma'^\lambda_{\nu\mu} W'_\lambda \\
&= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x'^\mu} W_\beta\right) - \frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\Gamma^\sigma_{\alpha\beta} W'_\lambda + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\alpha \partial x^\beta} W'_\lambda \\
&= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial^2 x^\beta}{\partial x^\alpha \partial x'^\mu} W_\beta + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial W_\beta}{\partial x^\alpha} - \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\Gamma^\sigma_{\alpha\beta}\left(\frac{\partial x'^\lambda}{\partial x^\sigma} W'_\lambda\right) + \frac{\partial x^\alpha}{\partial x'^\nu} W'_\lambda \frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial}{\partial x^\beta}\frac{\partial x'^\lambda}{\partial x^\alpha} \\
&= \frac{\partial x^\alpha}{\partial x'^\nu} W_\beta \frac{\partial}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x^\alpha} + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\left(\partial_\alpha W_\beta - \Gamma^\sigma_{\alpha\beta} W_\sigma\right) + \frac{\partial x^\alpha}{\partial x'^\nu} W'_\lambda \frac{\partial}{\partial x'^\mu}\frac{\partial x'^\lambda}{\partial x^\alpha} \\
&= \frac{\partial x^\alpha}{\partial x'^\nu} W_\beta \frac{\partial}{\partial x'^\mu}\delta^\beta_\alpha + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu} D_\alpha W_\beta + \frac{\partial x^\alpha}{\partial x'^\nu} W'_\lambda \frac{\partial}{\partial x^\alpha}\frac{\partial x'^\lambda}{\partial x'^\mu} \\
&= 0 + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu} D_\alpha W_\beta + \frac{\partial x^\alpha}{\partial x'^\nu} W'_\lambda \frac{\partial}{\partial x^\alpha}\delta^\lambda_\mu \\
&= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu} D_\alpha W_\beta + 0 \implies \text{covariant derivatives of covectors are tensors}
\end{aligned}
$$

Problem 6*

$$
\begin{aligned}
D_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} - \Gamma^\sigma_{\rho\nu} g_{\mu\sigma} = 0 \\
D_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0 \\
D_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - \Gamma^\sigma_{\nu\rho} g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu} g_{\rho\sigma} = 0
\end{aligned}
$$

$$
\begin{aligned}
0 &= D_\rho g_{\mu\nu} - D_\mu g_{\nu\rho} - D_\nu g_{\rho\mu} \\
&= \left(\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu}\right) + \left(\Gamma^\sigma_{\mu\nu} g_{\sigma\rho} + \Gamma^\sigma_{\nu\mu} g_{\rho\sigma}\right) + \left(\Gamma^\sigma_{\mu\rho} g_{\nu\sigma} - \Gamma^\sigma_{\rho\mu} g_{\sigma\nu}\right) + \left(\Gamma^\sigma_{\nu\rho} g_{\sigma\mu} - \Gamma^\sigma_{\rho\nu} g_{\mu\sigma}\right) \\
&= \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\sigma_{\mu\nu} g_{\sigma\rho} + 0 + 0 \qquad (g_{\alpha\beta} = g_{\beta\alpha},\ \text{torsion-free} \implies \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta})
\end{aligned}
$$

$$
\begin{aligned}
\implies \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} &= \frac{1}{2}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \\
\implies \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} g^{\lambda\rho} &= \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \\
\implies \Gamma^\lambda_{\mu\nu} &= \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)
\end{aligned}
$$