MAU22101: Group Theory
Assignment 4 due 30/11/2020

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Exercise 1

(i)

$$
\begin{aligned}
K &\subseteq S_4 && K \text{ is a subset of } S_4 \\
1_K &= 1 && K \text{ has an identity} \\
\text{By inspection, } k^{-1} &= k \ \forall\ k \in K && \text{each } k \in K \text{ has a unique inverse} \\
(12)(34)(13)(24) &= (14)(23) \\
(12)(34)(14)(23) &= (13)(24) \\
(13)(24)(12)(34) &= (14)(23) \\
(13)(24)(14)(23) &= (12)(34) \\
(14)(23)(12)(34) &= (13)(24) \\
(14)(23)(13)(24) &= (12)(34) \\
\implies xy &\in K \ \forall\ x, y \in K && \text{products of any } k \in K \text{ are also in } K
\end{aligned}
$$

Thus K is a subgroup of S_4.

(ii)

$$\begin{aligned}
|G| &= [G:H]\cdot|H| && \text{Lagrange's Theorem}\\
\implies [G:H] &= \frac{|G|}{|H|}\\
[S_4:K] &= \frac{|S_4|}{|K|}\\
&= \frac{24}{4}\\
[S_4:K] &= 6
\end{aligned}$$

$$\begin{aligned}
G_0 &= G - g_1H\\
&= S_4 - K && g_1 = 1\\
&= \{(34),(24),(23),(14),(13),(12),(234),(243),(134),(143),(124),\\
&\quad (142),(123),(132),(1234),(1243),(1324),(1342),(1423),(1432)\}\\
G_1 &= G_0 - g_2H\\
&= S_4 - K - (12)K && g_2 = (12)\\
&= S_4 - K - \{(12),(34),(1324),(1423)\}\\
&= \{(24),(23),(14),(13),(234),(243),(134),(143),(124),\\
&\quad (142),(123),(132),(1234),(1243),(1342),(1432)\}\\
G_2 &= G_1 - g_3H\\
&= G_1 - (24)K && g_3 = (24)\\
&= G_1 - \{(24),(1432),(13),(1234)\}\\
&= \{(23),(14),(234),(243),(134),(143),\\
&\quad (124),(142),(123),(132),(1243),(1342)\}\\
G_3 &= G_2 - g_4H\\
&= G_2 - (23)K && g_4 = (23)\\
&= G_2 - \{(23),(1342),(1243),(14)\}\\
&= \{(234),(243),(134),(143),(124),(142),(123),(132)\}\\
G_4 &= G_3 - g_5H\\
&= G_3 - (234)K && g_5 = (234)\\
&= G_3 - \{(234),(132),(143),(124)\}\\
&= \{(243),(134),(142),(123)\}\\
G_5 &= G_4 - g_6H\\
&= G_4 - (243)K && g_6 = (243)\\
&= G_4 - \{(243),(142),(123),(134)\}\\
&= \varnothing
\end{aligned}$$

$$\begin{aligned}
g_1K &= \{1,(12)(34),(13)(24),(14)(23)\}\\
g_2K &= \{(12),(34),(1324),(1423)\}\\
g_3K &= \{(24),(1432),(13),(1234)\}\\
g_4K &= \{(23),(1342),(1243),(14)\}\\
g_5K &= \{(234),(132),(143),(124)\}\\
g_6K &= \{(243),(142),(123),(134)\}\\
\{g_1,g_2,g_3,g_4,g_5,g_6\} &= \{1,(12),(24),(23),(234),(243)\}
\end{aligned}$$

(iii)

We must show that $\sigma K = K\sigma \ \forall \ \sigma \in S_4$. It is also sufficient to prove that $\sigma K \sigma^{-1} = K \ \forall \ \sigma \in S_4$.

$$\begin{aligned}
\sigma K \sigma^{-1} &= \left\{\sigma(1)\sigma^{-1}, \sigma(12)(34)\sigma^{-1}, \sigma(13)(24)\sigma^{-1}, \sigma(14)(23)\sigma^{-1}\right\} \\
&= \left\{1, \sigma(12)\sigma^{-1}\sigma(34)\sigma^{-1}, \sigma(13)\sigma^{-1}\sigma(24)\sigma^{-1}, \sigma(14)\sigma^{-1}\sigma(23)\sigma^{-1}\right\} \\
&= \{1, (\sigma(1)\sigma(2))(\sigma(3)\sigma(4)), (\sigma(1)\sigma(3))(\sigma(2)\sigma(4)), (\sigma(1)\sigma(4))(\sigma(2)\sigma(3))\} \text{ from } \textit{Exercise 2} \text{ of } \textit{Assignment 1}
\end{aligned}$$

$\sigma(1)$, $\sigma(2)$, $\sigma(3)$ and $\sigma(4)$ are all unique, and so $\sigma K \sigma^{-1}$ is simply the set including the identity and all possible unique products of two 2-cycles. However, K is simply the set including the identity and all possible unique products of two 2-cycles, and thus $\sigma K \sigma^{-1} = K \ \forall \ \sigma \in S_4$, i.e. $\sigma K = K\sigma \ \forall \ \sigma \in S_4$.

Say that σ_1, σ_2, …, σ_n are all represented by g_i, a left representative of K in S_4. This means that $\sigma_1 K = \sigma_2 K = \ldots = \sigma_n K = g_i K$. Since $\sigma K = K\sigma \ \forall \ \sigma \in S_4$, it follows that $K\sigma_1 = K\sigma_2 = \ldots = K\sigma_n = Kg_i$. Thus g_i must also be a right representative of K in S_4.

We have proven that $\sigma K = K\sigma \ \forall \ \sigma \in S_4$, which is simply the definition of K being a normal subgroup of S_4.

Exercise 2

(i)

$$\begin{aligned}
g, h, g^{-1} &\in G \\
\implies gHg^{-1} &\subseteq G && gHg^{-1} \text{ is a subset of } G \\
h = 1 \implies ghg^{-1} &= 1 = 1_{gHg^{-1}} && gHg^{-1} \text{ has an identity} \\
\left(gh_1g^{-1}\right)\left(gh_2g^{-1}\right) &= 1 \\
gh_1h_2g^{-1} &= 1 \\
h_1h_2 &= g^{-1}g \\
&= 1 \\
\implies h_2 &= {h_1}^{-1} \\
\implies \left(ghg^{-1}\right)^{-1} &= gh^{-1}g^{-1} && \text{each } ghg^{-1} \in gHg^{-1} \text{ has a unique inverse} \\
\left(gh_1g^{-1}\right)\left(gh_2g^{-1}\right) &= gh_1h_2g^{-1} \\
&= gh_3g^{-1}, \ h_3 = h_1h_2 \in H && \text{products of any } ghg^{-1} \in gHg^{-1} \text{ are also in } gHg^{-1}
\end{aligned}$$

Thus gHg^{-1} is a subgroup of G.

$$\begin{aligned}
&\text{Define } \pi : H \to gHg^{-1}, \ \pi(h) = ghg^{-1} \\
\pi(h_1h_2) &= gh_1h_2g^{-1} \\
&= gh_1g^{-1}gh_2g^{-1} \\
&= \pi(h_1)\pi(h_2) \implies \pi \text{ is a homomorphism} \\
h \in \ker(\pi) &\implies \pi(h) = 0 \\
&\implies ghg^{-1} = 0 \\
&\implies h = g^{-1}0g \\
&\implies h = 0 \\
\implies \ker(\pi) = \{0\} &\implies \pi \text{ is injective} \\
q \in gHg^{-1} &\implies q = ghg^{-1} \text{ for some } h. \\
h \in H &\implies \pi(h) = ghg^{-1} = q \\
&\implies \forall \ q \ \exists \ h \text{ such that } \pi(h) = q \implies \pi \text{ is surjective}
\end{aligned}$$

Thus gHg^{-1} is isomorphic to H. Therefore gHg^{-1} is a subgroup of G isomorphic to H.

(ii)

From *Exercise 2* of *Assignment 1*, $\tau\sigma\tau^{-1} = (\tau(i_1), \dots, \tau(i_j))$, where $\sigma = (i_1, \dots, i_j)$.
If $\sigma = ((i_1)(i_2))((i_3)(i_4)) = \sigma_1\sigma_2$ then $\tau\sigma\tau^{-1} = \tau\sigma_1\sigma_2\tau^{-1} = \tau\sigma_1\tau^{-1}\tau\sigma_2\tau^{-1} = (\tau(i_1)\tau(i_2))(\tau(i_3)\tau(i_4))$.
If $\sigma = 1$ then $\tau\sigma\tau^{-1} = \tau\tau^{-1} = 1 \;\forall\; \tau$. Thus we do not need to compute the conjugates of the identity, as it will always be equal to the identity itself.

If we consider $\sigma = (12)$, then $\tau(12)\tau^{-1} = (\tau(1)\tau(2))$. If we consider $\sigma = (34)$, then $\tau(34)\tau^{-1} = (\tau(3)\tau(4))$. Since $\tau \in S_4$, we can say that $\tau(1)$, $\tau(2)$, $\tau(3)$ and $\tau(4)$ are all unique, and thus $(\tau(1)\tau(2))$ and $(\tau(3)\tau(4))$ will share no elements. We therefore immediately know what $(\tau(3)\tau(4))$ is, given $(\tau(1)\tau(2))$.

$\tau(12)(34)\tau^{-1} = (\tau(1)\tau(2))(\tau(3)\tau(4))$. Thus, for any τ, $\tau K'\tau^{-1} = \{1, (\tau(1)\tau(2)), (\tau(3)\tau(4)), (\tau(1)\tau(2))(\tau(3)\tau(4))\}$. There are only 3 unique ways of writing a product of two 2-cycles, and so since $\tau(1)$, $\tau(2)$, $\tau(3)$ and $\tau(4)$ are all unique, there can only be a maximum of 3 conjugates of K'. It is therefore sufficient to show that if we can generate 3 unique sets $\tau K'\tau^{-1}$ for 3 different values for $\tau \in S_4$, then the conjugates of K' must be the 3 unique ways of writing a set of the identity, two 2-cycles, and a product of those two 2-cycles.

τ	$\tau(12)\tau^{-1} = (\tau(1)\tau(2))$	$\tau(34)\tau^{-1} = (\tau(3)\tau(4))$	$\tau(12)(34)\tau^{-1} = (\tau(1)\tau(2))(\tau(3)\tau(4))$
1	(12)	(34)	(12)(34)
(24)	(14)	(23)	(14)(23)
(23)	(13)	(24)	(13)(24)

Thus all conjugates of K' are $\{1, (12), (34), (12)(34)\} = K'$, $\{1, (14), (23), (14)(23)\}$ and $\{1, (13), (24), (13)(24)\}$.

(iii)

$$\begin{aligned} &(13)K' = \{(13), (123), (134), (1234)\} \\ &K'(13) = \{(13), (132), (143), (1432)\} \\ &(13)K' \neq K'(13) \\ \implies\; &\sigma K' = K'\sigma \text{ is not true } \forall\; \sigma \in S_4 \end{aligned}$$

Thus K' cannot be a normal subgroup of S_4.