MAU22101: Group Theory
Assignment 2 due 26/10/2020

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Exercise 1

$$\text{To prove} : 2^n \leqslant \begin{pmatrix} 2n \\ n \end{pmatrix} \; \forall \; n \in \mathbb{N}$$

$$\begin{aligned} n = 0 : 2^n &= 2^0 \\ &= 1 \\ \begin{pmatrix} 2n \\ n \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= 1 \\ \implies \text{True for } n &= 0 \end{aligned}$$

$$\begin{aligned} &\text{Assume true for } n = k, \\ &\text{i.e. } 2^k \leqslant \begin{pmatrix} 2k \\ k \end{pmatrix} \\ &\leqslant \frac{(2k)!}{k!(2k-k)!} \\ &\leqslant \frac{(2k)!}{(k!)^2} \\ 2^k &\leqslant \frac{(2k)(2k-1)\dots(k+1)}{k!} \end{aligned}$$

$$
\begin{aligned}
n = k+1 : 2^{k+1} &= 2\left(2^k\right) \\
&\leqslant 2\binom{2k}{k} \\
\binom{2k+2}{k+1} &= \frac{(2k+2)!}{(k+1)!(2k+2-(k+1))!)} \\
&= \frac{(2k+2)!}{((k+1)!)^2} \\
&= \frac{(2k+2)(2k+1)\ldots(k+2)}{(k+1)!} \\
&= \frac{(2k+2)(2k+1)}{(k+1)(k+1)}\cdot\frac{(2k)(2k-1)\ldots(k+1)}{k!} \\
&= \frac{(2k+2)(2k+1)}{(k+1)(k+1)}\binom{2k}{k} \\
&\geqslant 2\binom{2k}{k} \qquad \text{if } \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \geqslant 2
\end{aligned}
$$

$$
\begin{aligned}
4k^2+6k+2 &\geqslant 2\left(k^2+2k+1\right) \\
2k^2+2k &\geqslant 0 \\
k(k+1) &\geqslant 0 \\
k \leqslant -1 &\text{ or } k \geqslant 0 \\
k\in\mathbb{N} &\implies k \geqslant 0
\end{aligned}
$$

$$
\implies 2^{k+1} \leqslant \binom{2k+2}{k+1} \text{ if } 2^k \leqslant \binom{2k}{k}
$$

$\implies$ True for $n = k+1$ if true for $n = k$

$$
\begin{aligned}
&\text{True for } n = 0 \\
\implies &\text{True for } n = 1 \\
\implies &\text{True for } n = 2 \\
&\vdots \\
\implies &\text{True } \forall\, n \in \mathbb{N}, \\
&\text{i.e. } 2^n \leqslant \binom{2n}{n} \;\forall\, n\in\mathbb{N}
\end{aligned}
$$

Exercise 2

$$\begin{aligned} d \mid n \implies n &= ad,\ a \in \mathbb{Z} \\ 2^n - 1 &= 2^{ad} - 1 \\ &= \left(2^d\right)^a - 1 \\ &= \left(2^d - 1\right)\left(\left(2^d\right)^{a-1} + \left(2^d\right)^{a-2} + \ldots + 2^d + 1\right) \\ &= \left(2^d - 1\right) k \\ \implies 2^d - 1 &\mid 2^n - 1 \end{aligned}$$

$$\begin{aligned} 2^p - 1 \text{ is prime} &\implies 2^q - 1 \nmid 2^p - 1 \ \forall\ q \neq p, 1 \\ &\implies q \nmid p \ \forall\ q \neq p, 1 \\ &\implies p \text{ is prime} \end{aligned}$$

Exercise 3

$$\text{To prove : } \binom{2}{2} + \binom{3}{2} + \ldots + \binom{n}{2} = \binom{n+1}{3}$$

$$n = 0 : \binom{0}{2} = 0$$
$$\binom{1}{3} = 0$$
$$\implies \text{True for } n = 0$$

$$\text{Assume true for } n = k,$$
$$\text{i.e. } \binom{2}{2} + \binom{3}{2} + \ldots + \binom{k}{2} = \binom{k+1}{3}$$

$$n = k+1 : \binom{2}{2} + \binom{3}{2} + \ldots + \binom{k}{2} + \binom{k+1}{2} = \binom{k+1}{3} + \binom{k+1}{2}$$
$$= \binom{k+2}{3}$$
$$\implies \text{True for } n = k+1 \text{ if true for } n = k$$

$$\text{True for } n = 0$$
$$\implies \text{True for } n = 1$$
$$\implies \text{True for } n = 2$$
$$\vdots$$
$$\implies \text{True } \forall\, n \in \mathbb{N},$$
$$\text{i.e. } \binom{2}{2} + \binom{3}{2} + \ldots + \binom{n}{2} = \binom{n+1}{3}$$