

MAU22101: Group Theory
Assignment 2 due 26/10/2020

Ruaidhrí Campion
19333850
SF Theoretical Physics

I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at
<http://www.tcd.ie/calendar>.

I have completed the Online Tutorial in avoiding plagiarism ‘Ready, Steady, Write’, located at <http://tcd-ie.libguides.com/plagiarism/ready-steady-write>.

Exercise 1

$$\text{To prove : } 2^n \leq \binom{2n}{n} \quad \forall n \in \mathbb{N}$$

$$\begin{aligned} n = 0 : 2^n &= 2^0 \\ &= 1 \\ \binom{2n}{n} &= \binom{0}{0} \\ &= 1 \\ \implies \text{True for } n = 0 & \end{aligned}$$

Assume true for $n = k$,

$$\begin{aligned} \text{i.e. } 2^k &\leq \binom{2k}{k} \\ &\leq \frac{(2k)!}{k!(2k-k)!} \\ &\leq \frac{(2k)!}{(k!)^2} \\ 2^k &\leq \frac{(2k)(2k-1)\dots(k+1)}{k!} \end{aligned}$$

$$\begin{aligned}
n = k + 1 : 2^{k+1} &= 2(2^k) \\
&\leq 2 \binom{2k}{k} \\
\binom{2k+2}{k+1} &= \frac{(2k+2)!}{(k+1)!(2k+2-(k+1))!} \\
&= \frac{(2k+2)!}{((k+1)!)^2} \\
&= \frac{(2k+2)(2k+1)\dots(k+2)}{(k+1)!} \\
&= \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \cdot \frac{(2k)(2k-1)\dots(k+1)}{k!} \\
&= \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \binom{2k}{k} \\
&\geq 2 \binom{2k}{k} && \text{if } \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \geq 2 \\
&&& 4k^2 + 6k + 2 \geq 2(k^2 + 2k + 1) \\
&&& 2k^2 + 2k \geq 0 \\
&&& k(k+1) \geq 0 \\
&&& k \leq -1 \text{ or } k \geq 0 \\
&&& k \in \mathbb{N} \implies k \geq 0
\end{aligned}$$

$$\begin{aligned}
&\implies 2^{k+1} \leq \binom{2k+2}{k+1} \text{ if } 2^k \leq \binom{2k}{k} \\
&\implies \text{True for } n = k + 1 \text{ if true for } n = k
\end{aligned}$$

$$\begin{aligned}
&\text{True for } n = 0 \\
&\implies \text{True for } n = 1 \\
&\implies \text{True for } n = 2 \\
&\vdots \\
&\implies \text{True } \forall n \in \mathbb{N}, \\
&\text{i.e. } 2^n \leq \binom{2n}{n} \quad \forall n \in \mathbb{N}
\end{aligned}$$

Exercise 2

$$\begin{aligned} d \mid n &\implies n = ad, \quad a \in \mathbb{Z} \\ 2^n - 1 &= 2^{ad} - 1 \\ &= (2^d)^a - 1 \\ &= (2^d - 1) \left((2^d)^{a-1} + (2^d)^{a-2} + \dots + 2^d + 1 \right) \\ &= (2^d - 1) k \\ \implies 2^d - 1 &\mid 2^n - 1 \end{aligned}$$

$$\begin{aligned} 2^p - 1 \text{ is prime} &\implies 2^q - 1 \nmid 2^p - 1 \quad \forall q \neq p, 1 \\ &\implies q \nmid p \quad \forall q \neq p, 1 \\ &\implies p \text{ is prime} \end{aligned}$$

Exercise 3

$$\text{To prove : } \binom{2}{2} + \binom{3}{2} + \dots + \binom{n}{2} = \binom{n+1}{3}$$

$$\begin{aligned} n=0 : \binom{0}{2} &= 0 \\ \binom{1}{3} &= 0 \\ \implies \text{True for } n=0 & \end{aligned}$$

Assume true for $n = k$,

$$\text{i.e. } \binom{2}{2} + \binom{3}{2} + \dots + \binom{k}{2} = \binom{k+1}{3}$$

$$\begin{aligned} n=k+1 : \binom{2}{2} + \binom{3}{2} + \dots + \binom{k}{2} + \binom{k+1}{2} &= \binom{k+1}{3} + \binom{k+1}{2} \\ &= \binom{k+2}{3} \end{aligned}$$

\implies True for $n = k+1$ if true for $n = k$

True for $n = 0$

\implies True for $n = 1$

\implies True for $n = 2$

\vdots

\implies True $\forall n \in \mathbb{N}$,

$$\text{i.e. } \binom{2}{2} + \binom{3}{2} + \dots + \binom{n}{2} = \binom{n+1}{3}$$