MAU23403: Equations of Mathematical Physics
Homework 3 due 21/12/2020

Ruaidhrí Campion
19333850
SF Theoretical Physics

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Exercise 1

1.

$$\nabla \cdot f \equiv \frac{\partial f}{\partial x}\,\hat{x} + \frac{\partial f}{\partial y}\,\hat{y} + \frac{\partial f}{\partial z}\,\hat{z}$$
$$\nabla \cdot f = 7x^6 e^{-z^2}\,\hat{x} + 3y^2\,\hat{y} - 2x^7 z e^{-z^2}\,\hat{z}$$

2.

$$\begin{aligned}
\nabla \times \nabla g\,(x,y,z) &= \left(\frac{\partial g}{\partial x}\,\hat{x} + \frac{\partial g}{\partial y}\,\hat{y} + \frac{\partial g}{\partial z}\,\hat{z}\right) \times \left(\frac{\partial}{\partial x}\,\hat{x} + \frac{\partial}{\partial y}\,\hat{y} + \frac{\partial}{\partial z}\,\hat{z}\right) \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 g}{\partial y \partial z} - \frac{\partial^2 g}{\partial z \partial y}\right)\hat{x} - \left(\frac{\partial^2 g}{\partial x \partial z} - \frac{\partial^2 g}{\partial z \partial x}\right)\hat{y} + \left(\frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 g}{\partial y \partial x}\right)\hat{z} \\
\frac{\partial^2 a}{\partial b \partial c} &= \frac{\partial^2 a}{\partial c \partial b} \qquad \text{for a scalar field} \\
\implies \nabla \times \nabla g\,(x,y,z) &= 0\hat{x} - 0\hat{y} + 0\hat{z} \\
&= \vec{0}
\end{aligned}$$

3.

$$\nabla \times \vec{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}$$

(a)

$$\begin{aligned}
\nabla \times \vec{F} &= (0-0)\,\hat{x} + (0-0)\,\hat{y} + (-e^x \sin y - e^x \cos y)\,\hat{z} \\
&= -e^x\,(\sin y + \cos y)\,\hat{z} \\
&\neq \vec{0} \\
\implies \vec{F} &= e^x \cos y\,\hat{x} - e^x \sin y\,\hat{y} \text{ is not conservative}
\end{aligned}$$

(b)

$$\begin{aligned}\nabla \times \vec{F} &= (0-0)\,\hat{x} + \left(3z^2 - 3z^2\right)\hat{y} + (2y\cos x - 2y\cos x)\,\hat{z} \\ &= \vec{0} \\ \implies \vec{F} &= \left(y^2\cos x + z^3\right)\hat{x} + (2y\sin x - 4)\,\hat{y} + \left(3xz^2 + z\right)\hat{z} \text{ is conservative}\end{aligned}$$

$$\begin{aligned}\nabla\Phi &\equiv \frac{\partial\Phi}{\partial x}\,\hat{x} + \frac{\partial\Phi}{\partial y}\,\hat{y} + \frac{\partial\Phi}{\partial z}\,\hat{z} \\ \vec{F} &= \left(y^2\cos x + z^3\right)\hat{x} + (2y\sin x - 4)\,\hat{y} + \left(3xz^2 + z\right)\hat{z}\end{aligned}$$

$$\begin{aligned}\frac{\partial\Phi}{\partial x} &= y^2\cos x + z^3 \\ \Phi &= \int \left(y^2\cos x + z^3\right)dx \\ &= y^2\sin x + xz^3 + C_x \\ C_x &= -4y + \frac{z^2}{2}\end{aligned}$$

$$\begin{aligned}\frac{\partial\Phi}{\partial y} &= 2y\sin x - 4 \\ \Phi &= \int (2y\sin x - 4)\,dy \\ &= y^2\sin x - 4y + C_y \\ C_y &= xz^3 + \frac{z^2}{2}\end{aligned}$$

$$\begin{aligned}\frac{\partial\Phi}{\partial z} &= 3xz^2 + z \\ \Phi &= \int \left(3xz^2 + z\right)dz \\ &= xz^3 + \frac{z^2}{2} + C_z \\ C_z &= y^2\sin x - 4y\end{aligned}$$

$$\implies \Phi = xz^3 + y^2\sin x - 4y + \frac{z^2}{2}$$

Exercise 2

$$\int_C \vec{F}\cdot d\vec{\ell} = \int_t \vec{F}\Big|_C \cdot \frac{d\vec{r}}{dt}\,dt$$

1.

$$\begin{aligned}\vec{F}\Big|_C &= \left(x^2y\,\hat{x} + y\,\hat{y}\right)_C \\ &= \left(e^t\right)^2 e^{-t}\,\hat{x} + e^{-t}\,\hat{y} \\ &= e^t\,\hat{x} + e^{-t}\,\hat{y}\end{aligned}$$

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(e^t\,\hat{x} + e^{-t}\,\hat{y}\right) \\ &= e^t\,\hat{x} - e^{-t}\,\hat{y}\end{aligned}$$

$$\begin{aligned}\implies \int_C \vec{F}\cdot d\vec{\ell} &= \int_0^1 \left(e^t\,\hat{x} + e^{-t}\,\hat{y}\right)\cdot\left(e^t\,\hat{x} - e^{-t}\,\hat{y}\right)dt \\ &= \int_0^1 \left(e^{2t} - e^{-2t}\right)dt \\ &= \left.\frac{e^{2t} - e^{-2t}}{2}\right|_0^1 \\ &= \cosh(2t)|_0^1 \\ \int_C \vec{F}\cdot d\vec{\ell} &= \cosh(2) - 1\end{aligned}$$

2.

$$\vec{F}\Big|_C = \left(r^2\,\hat{x}\right)_C = \left(a^2\cos^2 t + a^2\sin^2 t + a^2t^2\right)\hat{x} = a^2\left(t^2+1\right)\hat{x}$$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(a\cos t\,\hat{x} + a\sin t\,\hat{y} + at\,\hat{z}\right) = a\left(-\sin t\,\hat{x} + \cos t\,\hat{y} + \hat{z}\right)$$

$$\implies \int_C \vec{F}\cdot d\vec{\ell} = \int_0^{\frac{\pi}{2}} a^2\left(t^2+1\right)\hat{x}\cdot a\left(-\sin t\,\hat{x} + \cos t\,\hat{y} + \hat{z}\right)dt = -a^3\left(\int_0^{\frac{\pi}{2}} t^2\sin t\,dt + \int_0^{\frac{\pi}{2}}\sin t\,dt\right)$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} t^2\sin t\,dt &= \left(-t^2\cos t + \int 2t\cos t\,dt\right)_0^{\frac{\pi}{2}} && u = t^2,\; dv = \sin t\,dt\\ &= \left(-t^2\cos t + 2\left(t\sin t - \int \sin t\,dt\right)\right)_0^{\frac{\pi}{2}} && u = t,\; dv = \cos t\,dt\\ &= \left(-t^2\cos t + 2t\sin t + 2\cos t\right)_0^{\frac{\pi}{2}}\\ &= -\frac{\pi^2}{4}\cos\frac{\pi}{2} + \pi\sin\frac{\pi}{2} + 2\cos\frac{\pi}{2} + 0 - 0 - 2\cos 0\\ &= \pi - 2\end{aligned}$$

$$\int_0^{\frac{\pi}{2}}\sin t\,dt = -\left(\cos\frac{\pi}{2} - \cos 0\right) = 1$$

$$\implies \int_C \vec{F}\cdot d\vec{\ell} = -a^3\left(\pi - 2 + 1\right)$$

$$\int_C \vec{F}\cdot d\vec{\ell} = a^3\left(1-\pi\right)$$

3.

$$\vec{F}\Big|_C = \left(x^2+y^2\right)^{-\frac{3}{2}}\left(x\,\hat{x} + y\,\hat{y}\right)\Big|_C = \left(e^{2t}\sin^2 t + e^{2t}\cos^2 t\right)^{-\frac{3}{2}}\left(e^t\sin t\,\hat{x} + e^t\cos t\,\hat{y}\right) = e^{-3t}\left(e^t\sin t\,\hat{x} + e^t\cos t\,\hat{y}\right) = e^{-2t}\left(\sin t\,\hat{x} + \cos t\,\hat{y}\right)$$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(e^t\sin t\,\hat{x} + e^t\cos t\,\hat{y}\right) = \left(e^t\sin t + e^t\cos t\right)\hat{x} + \left(e^t\cos t - e^t\sin t\right)\hat{y} = e^t\left(\left(\cos t + \sin t\right)\hat{x} + \left(\cos t - \sin t\right)\hat{y}\right)$$

$$\begin{aligned}\implies \int_C \vec{F}\cdot d\vec{\ell} &= \int_0^1 e^{-2t}\left(\sin t\,\hat{x} + \cos t\,\hat{y}\right)\cdot e^t\left(\left(\cos t + \sin t\right)\hat{x} + \left(\cos t - \sin t\right)\hat{y}\right)dt\\ &= \int_0^1 e^{-t}\left(\sin t\cos t + \sin^2 t + \cos^2 t - \sin t\cos t\right)dt\\ &= \int_0^1 e^{-t}\,dt\\ &= -e^{-1} + e^0\end{aligned}$$

$$\int_C \vec{F}\cdot d\vec{\ell} = 1 - \frac{1}{e}$$

4.

$$\vec{r}(t) = 2\cos t\,\hat{x} + 2\sin t\,\hat{y}$$

$$\begin{aligned} \vec{F}\Big|_C &= \left(x^3\,\hat{x} + xy\,\hat{y}\right)_C \\ &= 8\cos^3 t\,\hat{x} + 4\sin t\cos t\,\hat{y} \end{aligned} \qquad \begin{aligned} \frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(2\cos t\,\hat{x} + 2\sin t\,\hat{y}\right) \\ &= -2\sin t\,\hat{x} + 2\cos t\,\hat{y} \end{aligned}$$

$$\begin{aligned} \implies \int_C \vec{F}\cdot d\vec{\ell} &= \int_0^\pi \left(8\cos^3 t\,\hat{x} + 4\sin t\cos t\,\hat{y}\right)\cdot\left(-2\sin t\,\hat{x} + 2\cos t\,\hat{y}\right) dt \\ &= \int_0^\pi \left(-16\sin t\cos^3 t + 8\sin t\cos^2 t\right) dt \\ &= 8\int_{u(0)}^{u(\pi)} \left(2u^3 - u^2\right) du \qquad\qquad \begin{aligned} u &= \cos t \\ du &= -\sin t\, dt \end{aligned} \\ &= 8\left(\frac{u^4}{2} - \frac{u^3}{3}\right)_{u(0)}^{u(\pi)} \\ &= 8\left(\frac{\cos^4 t}{2} - \frac{\cos^3 t}{3}\right)_0^\pi \\ &= 8\left(\frac{\cos^4\pi - \cos^4 0}{2} - \frac{\cos^3\pi - \cos^3 0}{3}\right) \end{aligned}$$

$$\int_C \vec{F}\cdot d\vec{\ell} = \frac{16}{3}$$

Exercise 3

In this exercise, it is assumed that the given hemisphere is closed by a disc of radius 1 on the xy-plane.

$$\text{Gauss's Theorem} \implies \oint_S \vec{F}\cdot d\vec{S} = \int_V \nabla\cdot\vec{F}\, dV$$

$$\begin{aligned} \nabla\cdot\vec{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &= 3x^2 + 3y^2 + 3z^2 \\ &= 3r^2 \end{aligned} \qquad \begin{aligned} dV &= dx\,dy\,dz \\ &= r^2\sin\theta\, dr\, d\theta\, d\phi \end{aligned}$$

$$\begin{aligned} \text{Flux} &\equiv \oint_S \vec{F}\cdot d\vec{S} \\ &= \int_V \nabla\cdot\vec{F}\, dV \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 3r^2\, r^2\sin\theta\, dr\, d\theta\, d\phi \\ &= 3\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^1 r^4\, dr \\ &= 3\,(2\pi - 0)\left(-\cos\frac{\pi}{2} + \cos 0\right)\left(\frac{1^5}{5} - 0\right) \end{aligned}$$

$$\text{Flux} = \frac{6\pi}{5}$$

Since any vector from the origin will be perpendicular to $d\vec{S}$ for the disc, the flux above is simply the flux through the curved part of the hemisphere. Thus, the flux through a whole sphere would be twice this, i.e. $\frac{12\pi}{5}$.

Exercise 4

1.

$$
\begin{aligned}
\text{Flux} &= \oint_S \vec{F} \cdot d\vec{S} \\
&= \oint_S \left.\frac{\vec{r}}{r^3}\right|_S \cdot d\vec{S} \\
&= \oint_S \left.\frac{\hat{r}}{r^2}\right|_S \cdot d\vec{S} \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{\hat{r}}{a^2} \cdot a^2 \sin\theta \, d\theta \, d\phi \, \hat{r} \\
&= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \\
&= (2\pi - 0)\,(\cos\pi + \cos 0) \\
\text{Flux} &= 4\pi
\end{aligned}
$$

2.

The origin $(0, 0, 0)$ is not contained within this box, and thus the net flux out of the box is 0.

3.

Consider removing a sphere of radius < 1 centred at the origin from this box. Since the origin is no longer contained in this new region, the net flux through it is 0. From part 1, the net flux out of a sphere for this $\vec{F}$ is 4π, for any radius. Thus, the flux flowing into (but not out of) the new region is -4π. Since the net flux through the region is 0, the flux flowing out of the region must be 4π. This flux flowing out of this region is the same as the net flux flowing out of the given box, and is thus 4π.

Exercise 5

$$
\begin{aligned}
C \text{ is a closed path} \implies & \int_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r} \\
\text{Stoke's Theorem} \implies & \oint_C \vec{F} \cdot d\vec{r} = \int_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} \\
& \nabla \times \vec{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z} \\
& = (\sin y - \sin y)\,\hat{x} + (36x - 36x)\,\hat{y} + (6\cos x - 6\cos x)\,\hat{z} \\
& = \vec{0} \\
\implies & \int_C \vec{F} \cdot d\vec{r} = \int_S \vec{0} \cdot d\vec{S} \\
& \int_C \vec{F} \cdot d\vec{r} = 0
\end{aligned}
$$