MAU23403: Equations of Mathematical Physics
Homework 2 due 26/11/2020

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Exercise 1

$$
\begin{aligned}
f(x) &= xe^{-(x-1)^2} \\
\tilde{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-ikx}\, xe^{-(x-1)^2} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, e^{-ik(1+u)}\, (1+u)\, e^{-u^2} && u = x-1 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, e^{-ik}\, e^{-iku}\left(e^{-u^2} + ue^{-u^2}\right) \\
&= e^{-ik}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(du\, e^{-u^2}\, e^{-iku}\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(du\, ue^{-u^2}\, e^{-iku}\right)\right) \\
&= e^{-ik}\left(\mathcal{F}\left(e^{-u^2}\right) + \mathcal{F}\left(ue^{-u^2}\right)\right) \\
&= e^{-ik}\left(\mathcal{F}\left(e^{-u^2}\right) + \mathcal{F}\left(-\frac{1}{2}\frac{d}{du}e^{-u^2}\right)\right) \\
&= e^{-ik}\left(\mathcal{F}\left(e^{-u^2}\right) - \frac{ik}{2}\mathcal{F}\left(e^{-u^2}\right)\right) && \mathcal{F}(f'(x)) = ik\,\mathcal{F}(f(x)) \\
&= \frac{2-ik}{2}\, e^{-ik}\, \mathcal{F}\left(e^{-u^2}\right) \\
&= \frac{2-ik}{2}\, e^{-ik}\, \frac{e^{-\frac{k^2}{4}}}{2\sqrt{\pi}} && \mathcal{F}\left(e^{-\alpha x^2}\right) = \frac{e^{-\frac{k^2}{4}}}{2\sqrt{\pi\alpha}} \\
\tilde{f}(k) &= \frac{2-ik}{4\sqrt{\pi}}\, e^{-\left(\frac{k^2}{4}+ik\right)}
\end{aligned}
$$

Exercise 2

(1)

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, e^x\, \delta(x+1) &= \int_{-\infty}^{\infty} dx\, e^x\, \delta(x-(-1)) \\ &= e^x\,|_{-1} \\ &= e^{-1}\end{aligned}$$

$$\int_{-\infty}^{\infty} dx\, e^x\, \delta(x+1) = \frac{1}{e}$$

(2)

$$\begin{aligned} {x_i}^2 - 3x_i + 2 &= 0 \\ \implies x_1 = 2,\ x_2 &= 1,\ i = 1,2 \\ \left(x^2 - 3x + 2\right)'\Big|_{x_i} &= 2x_i - 3 \\ &= \pm 1 \\ \delta\left(x^2 - 3x + 2\right) &= \sum_{i=1}^{n} \frac{\delta(x-x_i)}{\left|(x^2-3x+2)'\right|_{x_i}} \\ &= \frac{\delta(x-2)}{|1|} + \frac{\delta(x-1)}{|-1|} \\ &= \delta(x-2) + \delta(x-1) \\ \implies \int_{-3}^{0} dx\, \delta\left(x^2 - 3x + 2\right) &= \int_{-3}^{0} dx\, \delta(x-2) + \int_{-3}^{0} dx\, \delta(x-1) \\ &= \int_{-5}^{2} da\, \delta(a) + \int_{-4}^{-1} db\, \delta(b) \end{aligned}$$

$$\int_{-3}^{0} dx\, \delta\left(x^2 - 3x + 2\right) = 0$$

(3)

$$\begin{aligned}\int_{-\infty}^{\infty} dx\ \cos x\, \delta'(x) &= -(\cos x)'|_0 \\ &= \sin x|_0\end{aligned}$$

$$\int_{-\infty}^{\infty} dx\ \cos x\, \delta'(x) = 0$$

(4)

$$
\begin{aligned}
\sin\left(\frac{1}{x_i}\right) &= 0 \\
\implies x_i &= \frac{1}{i\pi},\ i \in \mathbb{Z}^* \\
\left(\sin\left(\frac{1}{x}\right)\right)'\Bigg|_{x_i} &= \cos\left(\frac{1}{x_i}\right)\frac{1}{{x_i}^2} \\
&= \cos(i\pi)\,(i\pi)^2 \\
&= \pm i^2\pi^2 \\
\delta\left(\sin\left(\frac{1}{x}\right)\right) &= \sum_i \frac{\delta(x - x_i)}{\left|\left(\sin\left(\frac{1}{x}\right)\right)'\right|_{x_i}} \\
&= \sum_{i\in\mathbb{Z}^*} \frac{\delta\left(x - \frac{1}{i\pi}\right)}{i^2\pi^2} \\
\implies \int_0^1 dx\,\delta\left(\sin\left(\frac{1}{x}\right)\right) &= \int_0^1 dx \sum_{i\in\mathbb{Z}^*} \frac{\delta\left(x - \frac{1}{i\pi}\right)}{i^2\pi^2} \\
&= \frac{1}{\pi^2}\sum_{i=1}^{\infty}\frac{1}{i^2}\int_0^1 dx\,\delta\left(x - \frac{1}{i\pi}\right) && \text{integral vanishes } \forall\ i < 0 \\
&= \frac{1}{\pi^2}\sum_{i=1}^{\infty}\frac{1}{i^2}\int_{-\frac{1}{i\pi}}^{1-\frac{1}{i\pi}} du_i\,\delta(u_i) && u_i = x - \frac{1}{i\pi} \\
0 < \frac{1}{i\pi} < 1\ \forall\ i > 0 &\implies -\frac{1}{i\pi} < 0 < 1 - \frac{1}{i\pi}\ \forall\ i > 0 \\
\implies \int_0^1 dx\,\delta\left(\sin\left(\frac{1}{x}\right)\right) &= \frac{1}{\pi^2}\sum_{i=1}^{\infty}\frac{1}{i^2} \\
&= \frac{1}{\pi^2}\,\frac{\pi^2}{6} \\
\int_0^1 dx\,\delta\left(\sin\left(\frac{1}{x}\right)\right) &= \frac{1}{6}
\end{aligned}
$$

Exercise 3

(5)

$$h(x) = e^{-\alpha x^2}$$

$$\tilde{h}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}\, e^{-ikx}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 - ikx}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-\alpha\left(x+\frac{ik}{2\alpha}\right)^2 + \frac{(ik)^2}{4\alpha}} \qquad \text{completing the square}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, e^{-\alpha u^2} e^{-\frac{k^2}{4\alpha}} \qquad u = x + \frac{ik}{2\alpha}$$

$$= e^{-\frac{k^2}{4\alpha}}\frac{1}{2\pi}\int_{-\infty}^{\infty} du\, e^{-\alpha u^2}$$

$$\tilde{h}(k) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{k^2}{4\alpha}} \qquad \text{Gaussian integral: } \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

(6)

$$p(x) = e^{-\alpha(x-b)^2}$$

$$\tilde{p}(k) = \int_{-\infty}^{\infty} dx\, e^{-\alpha(x-b)^2}\, e^{-ikx}$$

$$= \int_{-\infty}^{\infty} du\, e^{-\alpha u^2}\, e^{-ik(u+b)} \qquad u = x - b$$

$$= e^{-ikb}\int_{-\infty}^{\infty} du\, e^{-\alpha u^2}\, e^{-iku}$$

$$= e^{-ikb}\,\tilde{h}(k)$$

$$\tilde{p}(k) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\left(\frac{k^2}{4\alpha}+ikb\right)}$$

(7)

$$
\begin{aligned}
f(x,y,z) &= x^2\, e^{-x^2} e^{-y^2} e^{-z^2} \\
\tilde{f}(k,l,m) &= \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} dy\, e^{-ily} \int_{-\infty}^{\infty} dz\, e^{-imz}\, f(x,y,z) \\
&= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, x^2\, e^{-x^2}\, e^{-ikx}\right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, e^{-y^2}\, e^{-ily}\right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dz\, e^{-z^2}\, e^{-imz}\right) \\
&= \mathcal{F}_k\Big(x^2\, e^{-x^2}\Big)\, \mathcal{F}_l\Big(e^{-y^2}\Big)\, \mathcal{F}_m\Big(e^{-z^2}\Big) \\
\mathcal{F}_l\Big(e^{-y^2}\Big) &= \frac{e^{-\frac{l^2}{4}}}{2\sqrt{\pi}} \\
\mathcal{F}_m\Big(e^{-z^2}\Big) &= \frac{e^{-\frac{m^2}{4}}}{2\sqrt{\pi}} \\
\mathcal{F}_k\Big(x^2\, e^{-x^2}\Big) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, x^2\, e^{-x^2}\, e^{-ikx} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, x^2\, e^{-x^2 - ikx} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, x^2\, e^{-\left(x + \frac{ik}{2}\right)^2 + \frac{i^2 k^2}{4}} \\
&= \frac{e^{-\frac{k^2}{4}}}{2\pi} \int_{-\infty}^{\infty} dx\, x^2\, e^{-\left(x + \frac{ik}{2}\right)^2} \\
&= \frac{e^{-\frac{k^2}{4}}}{2\pi} \int_{-\infty}^{\infty} du\, \left(u - \frac{ik}{2}\right)^2 e^{-u^2} \qquad\qquad u = x + \frac{ik}{2} \\
&= \frac{e^{-\frac{k^2}{4}}}{2\pi} \left(\int_{-\infty}^{\infty} du\, u^2\, e^{-u^2} - ik \int_{-\infty}^{\infty} du\, u\, e^{-u^2} - \frac{k^2}{4} \int_{-\infty}^{\infty} du\, e^{-u^2}\right) \\
&= \frac{e^{-\frac{k^2}{4}}}{2\pi} \left(I_1 - ik\, I_2 - \frac{k^2}{4} I_3\right)
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} da\, a^2\, e^{-a^2} \\
&= \int_{-\infty}^{\infty} da\, a \left(-\frac{1}{2}\frac{d}{da} e^{-a^2}\right) \\
&= -\frac{a}{2} e^{-a^2}\Big|_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} da\, e^{-a^2} && u = a \\
& && dv = -\frac{1}{2}\frac{d}{da} e^{-a^2} \\
&= 0 + \frac{\sqrt{\pi}}{2} \\
&= \frac{\sqrt{\pi}}{2} \\
I_2 &= \int_{-\infty}^{\infty} db\, b\, e^{-b^2} \\
&= \int_{-\infty}^{\infty} db \left(-\frac{1}{2}\frac{d}{db} e^{-b^2}\right) \\
&= -\frac{e^{-b^2}}{2}\Bigg|_{-\infty}^{\infty} \\
&= 0 \\
I_3 &= \int_{-\infty}^{\infty} dc\, e^{-c^2} \\
&= \sqrt{\pi}
\end{aligned}
$$

$$
\begin{aligned}
\implies \mathcal{F}_k\left(x^2\, e^{-x^2}\right) &= \frac{e^{-\frac{k^2}{4}}}{2\pi}\left(\frac{\sqrt{\pi}}{2} - ik\,(0) - \frac{k^2\sqrt{\pi}}{4}\right) \\
&= \left(2 - k^2\right) \frac{e^{-\frac{k^2}{4}}}{8\sqrt{\pi}}
\end{aligned}
$$

$$
\implies \tilde{f}(k,l,m) = \left(2 - k^2\right) \frac{e^{-\frac{k^2}{4}}}{8\sqrt{\pi}} \frac{e^{-\frac{l^2}{4}}}{2\sqrt{\pi}} \frac{e^{-\frac{m^2}{4}}}{2\sqrt{\pi}}
$$

$$
\tilde{f}(k,l,m) = \frac{2 - k^2}{32\pi^{\frac{3}{2}}}\, e^{-\frac{1}{4}\left(k^2 + l^2 + m^2\right)}
$$

(8)

$$
\begin{aligned}
g(x,y) &= \delta(x)\,\delta\big(y^2 - {y_0}^2\big) \\
\tilde{g}(k,l) &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} dy\, e^{-ily}\, g(x,y) \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, \delta(x)\, e^{-ikx}\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, \delta\big(y^2 - {y_0}^2\big)\, e^{-ily}\right) \\
&= \mathcal{F}_k(\delta(x))\, \mathcal{F}_l\big(\delta\left(y^2 - {y_0}^2\right)\big) \\
\mathcal{F}_k(\delta(x)) &= \frac{1}{2\pi} \\
{y_i}^2 - {y_0}^2 &= 0 \\
\implies y_1 = y_0,\ y_2 &= -y_0,\ i = 1,2 \\
\big(y^2 - {y_0}^2\big)'\Big|_{y_i} &= 2y_i \\
\delta\big(y^2 - {y_0}^2\big) &= \sum_{i=1}^{n} \frac{\delta(y - y_i)}{\big|(y^2 - {y_0}^2)'\big|_{y_i}} \\
&= \frac{\delta(y - y_0)}{|2y_0|} + \frac{\delta(y + y_0)}{|-2y_0|} \\
&= \frac{\delta(y - y_0) + \delta(y + y_0)}{2|y_0|} \\
\implies \mathcal{F}_l\big(\delta\big(y^2 - {y_0}^2\big)\big) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \left(\frac{\delta(y - y_0) + \delta(y + y_0)}{2|y_0|}\right) e^{-ily} \\
&= \frac{1}{4\pi|y_0|}\left(\int_{-\infty}^{\infty} dy\, \delta(y - y_0)\, e^{-ily} + \int_{-\infty}^{\infty} dy\, \delta(y + y_0)\, e^{-ily}\right) \\
&= \frac{2\pi}{4\pi|y_0|}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, \delta(p)\, e^{-il(p+y_0)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, \delta(q)\, e^{-il(q-y_0)}\right) \qquad \begin{aligned} p &= y - y_0 \\ q &= y + y_0 \end{aligned} \\
&= \frac{1}{2|y_0|}\left(\frac{e^{-ily_0}}{2\pi}\int_{-\infty}^{\infty} dp\, \delta(p)\, e^{-ilp} + \frac{e^{ily_0}}{2\pi}\int_{-\infty}^{\infty} dq\, \delta(q)\, e^{-ilq}\right) \\
&= \frac{e^{ily_0} + e^{-ily_0}}{2|y_0|}\, \mathcal{F}(\delta(r)) \\
&= \frac{\cos(ly_0)}{2\pi|y_0|} \\
\implies \tilde{g}(k,l) &= \frac{1}{2\pi}\,\frac{\cos(ly_0)}{2\pi|y_0|} \\
\tilde{g}(k,l) &= \frac{\cos(ly_0)}{4\pi^2|y_0|}
\end{aligned}
$$