MAU23403: Equations of Mathematical Physics
Homework 1 due 26/10/2020

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Exercise 1

$$\begin{aligned}
a_0 &= \frac{2}{l}\int_0^l f(x)\,dx \\
&= \frac{2}{2\pi}\int_0^{2\pi} (\pi - t)^2\,dt \\
&= \frac{1}{\pi}\int_0^{2\pi} \left(\pi^2 - 2\pi t + t^2\right) dt \\
&= \frac{1}{\pi}\left(\pi^2 t - \pi t^2 + \frac{t^3}{3}\bigg|_0^{2\pi}\right) \\
&= 2\pi^2 - 4\pi^2 + \frac{8\pi^2}{3} \\
&= \frac{2\pi^2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{2\pi n x}{l}\right) dx \\
&= \frac{2}{2\pi}\int_0^{2\pi} (\pi - t)^2 \cos\left(\frac{2\pi n t}{2\pi}\right) dt \\
&= \frac{1}{\pi}\int_0^{2\pi} (\pi - t)^2 \cos(nt)\,dt \\
&= \frac{1}{\pi}\left((\pi - t)^2 \frac{\sin(nt)}{n}\bigg|_0^{2\pi} + \int_0^{2\pi} 2(\pi - t)\frac{\sin(nt)}{n}\,dt\right) && u = (\pi - t)^2,\ dv = \cos(nt)\,dt \\
&= \frac{1}{\pi}\left((\pi - t)^2 \frac{\sin(nt)}{n}\bigg|_0^{2\pi} + \frac{2}{n}\left(-(\pi - t)\frac{\cos(nt)}{n}\bigg|_0^{2\pi} - \int_0^{2\pi} \frac{\cos(nt)}{n}\,dt\right)\right) && u = \pi - t,\ dv = \sin(nt)\,dt \\
&= \frac{1}{\pi}\left((\pi - t)^2 \frac{\sin(nt)}{n}\bigg|_0^{2\pi} - \frac{2}{n^2}\left((\pi - t)\cos(nt)\big|_0^{2\pi} + \frac{\sin(nt)}{n}\bigg|_0^{2\pi}\right)\right) \\
&= \frac{1}{\pi}\left(0 - \frac{2}{n^2}(-2\pi) + 0\right) \\
&= \frac{4}{n^2}
\end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{2}{l}\int_0^l f(x)\sin\left(\frac{2\pi n x}{l}\right)dx \\
&= \frac{2}{2\pi}\int_0^{2\pi}(\pi-t)^2\sin\left(\frac{2\pi n t}{2\pi}\right)dt \\
&= \frac{1}{\pi}\int_0^{2\pi}(\pi-t)^2\sin(nt)\,dt \\
&= \frac{1}{\pi}\left(-(\pi-t)^2\frac{\cos(nt)}{n}\bigg|_0^{2\pi} - \int_0^{2\pi}2(\pi-t)\frac{\cos(nt)}{n}\,dt\right) && u=(\pi-t)^2,\ dv=\sin(nt)\,dt \\
&= \frac{1}{\pi}\left(-(\pi-t)^2\frac{\cos(nt)}{n}\bigg|_0^{2\pi} - \frac{2}{n}\left((\pi-t)\frac{\sin(nt)}{n}\bigg|_0^{2\pi} + \int_0^{2\pi}\frac{\sin(nt)}{n}\,dt\right)\right) && u=\pi-t,\ dv=\cos(nt)\,dt \\
&= \frac{1}{\pi}\left(-(\pi-t)^2\frac{\cos(nt)}{n}\bigg|_0^{2\pi} - \frac{2}{n^2}\left((\pi-t)\sin(nt)|_0^{2\pi} - \frac{cos(nt)}{n}\bigg|_0^{2\pi}\right)\right) \\
&= \frac{1}{\pi}\left(0-\frac{2}{n^2}(0-0)\right) \\
&= 0
\end{aligned}
$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi n}{l}x\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{2\pi n}{l}x\right)$$

$$f(t) = (\pi-t)^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4\cos(nt)}{n^2}$$

$$
\begin{aligned}
f(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4\cos(n\pi)}{n^2} &= (\pi-\pi)^2 \\
-4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} &= \frac{\pi^2}{3} \\
\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} &= \frac{\pi^2}{12}
\end{aligned}
$$

Exercise 2

1.

$$
\begin{aligned}
a_0 &= \frac{2}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(x)\,dx \\
&= \frac{2}{2\pi}\left(\int_{-\pi}^{0} 0 + \int_{0}^{\pi} \sin x\,dx\right) \\
&= -\frac{1}{\pi}\cos x\Big|_0^\pi \\
&= \frac{2}{\pi}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(x)\cos\left(\frac{2\pi n x}{l}\right)dx \\
&= \frac{2}{2\pi}\left(\int_{-\pi}^{0} 0 + \int_{0}^{\pi} \sin x\cos(nx)\,dx\right) \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin x\cos(nx)\,dx && (1) \\
&= \frac{1}{\pi}\left(-\cos x\cos(nx)|_0^\pi - \int_0^\pi n\sin(nx)\cos x\,dx\right) && u=\cos(nx),\ dv=\sin x\,dx \\
&= -\frac{1}{\pi}\left(\cos x\cos(nx)|_0^\pi + n\left(\sin x\sin(nx)|_0^\pi - \int_0^\pi n\sin x\cos(nx)\,dx\right)\right) && u=\sin(nx),\ dv=\cos x\,dx \\
&= -\frac{1}{\pi}\left(\cos x\cos(nx)|_0^\pi + n\,\sin x\sin(nx)|_0^\pi - n^2\int_0^\pi \sin x\cos(nx)\,dx\right) && (2) \\
\text{Let } I_a &= \int_0^\pi \sin x\cos(nx)\,dx \\
a_n = \frac{1}{\pi}I_a &= -\frac{1}{\pi}\left(\cos x\cos(nx)|_0^\pi + n\sin x\sin(nx)|_0^\pi - n^2 I_a\right) \\
I_a &= \frac{\cos x\cos(nx) + n\sin x\sin(nx)}{n^2-1}\Big|_0^\pi \\
&= \frac{-\cos(n\pi)-1}{n^2-1} \\
\implies a_n &= -\frac{1}{\pi}\frac{(-1)^n+1}{n^2-1},\ n\geqslant 2 \\
a_1 &= \frac{1}{\pi}\int_0^\pi \sin x\cos x\,dx \\
&= \frac{1}{\pi}\int_0^\pi \frac{\sin(2x)}{2} \\
&= \frac{1}{2\pi}\frac{-\cos(2x)}{2}\Big|_0^\pi \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{2}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(x)\sin\left(\frac{2\pi n x}{l}\right)dx \\
&= \frac{2}{2\pi}\left(\int_{-\pi}^{0} 0 + \int_0^{\pi} \sin x \sin(nx)\,dx\right) \\
&= \frac{1}{\pi}\int_0^{\pi} \sin x \sin(nx)\,dx \qquad (3)\\
&= \frac{1}{\pi}\left(-\cos x \sin(nx)|_0^{\pi} + \int_0^{\pi} n\cos x\cos(nx)\,dx\right) \qquad u=\sin(nx),\ dv=\sin x\,dx \\
&= \frac{1}{\pi}\left(-\cos x\sin(nx)|_0^{\pi} + n\left(\sin x\cos(nx)|_0^{\pi} + \int_0^{\pi} n\sin x\sin(nx)\,dx\right)\right) \\
&= \frac{n^2}{\pi}\int_0^{\pi}\sin x\sin(nx)\,dx \qquad (4)\\
\text{Let } I_b &= \int_0^{\pi}\sin x\sin(nx)\,dx \\
b_n = \frac{1}{\pi}I_b &= \frac{n^2}{\pi}I_b \\
I_b &= n^2 I_b \\
\implies I_b &= 0\ \forall\ n \geqslant 2 \\
\implies b_n &= 0\ \forall\ n \geqslant 2 \\
b_1 &= \frac{1}{\pi}\int_0^{\pi}\sin^2 x\,dx \\
&= \frac{1}{2\pi}\left(x - \frac{\sin(2x)}{2}\right)\bigg|_0^{\pi} \\
&= \frac{1}{2}
\end{aligned}
$$

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi n}{l}x\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{2\pi n}{l}x\right)
$$

$$
f(x) = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n+1}{n^2-1}\cos(nx) + \frac{\sin x}{2}
$$

The $f(x)$ we are given is defined in terms of a single $\sin x$, and so it makes sense that the Fourier series only has one $\sin x$ and no other sin, i.e. that $b_n = 0$ for any $n \geqslant 1$.

2.

$$
\begin{aligned}
f(0) = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n+1}{n^2-1} &= 0 \\
\sum_{n=2}^{\infty}\frac{(-1)^n+1}{n^2-1} &= 1 \\
\frac{2}{2^2-1} + 0 + \frac{2}{4^2-1} + 0 + \frac{2}{6^2-1} + \ldots &= 1 \\
\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \ldots &= \frac{1}{2}
\end{aligned}
$$

Exercise 3

1.

$$F(x) = \begin{cases} f(-x), & -l < x < 0 \\ f(x), & 0 < x < l \end{cases}$$

$$F(t) = \begin{cases} -t, & -\dfrac{\pi}{2} < t < 0 \\ t, & 0 < t < \dfrac{\pi}{2} \end{cases}$$

$$\begin{aligned} a_0 &= \frac{2}{l} \int_0^l f(x)\,dx \\ &= \frac{2}{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} t\,dt \\ &= \frac{4}{\pi} \frac{t^2}{2} \bigg|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{2}{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} t \cos\left(\frac{n\pi t}{\frac{\pi}{2}}\right) dt \\ &= \frac{4}{\pi} \int_0^{\frac{\pi}{2}} t \cos(2nt)\,dt \\ &= \frac{4}{\pi} \left(\frac{t \sin(2nt)}{2n} \bigg|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin(2nt)}{2n}\,dt \right) \\ &= \frac{4}{\pi} \left(0 + \frac{\cos(2nt)}{(2n)^2} \bigg|_0^{\frac{\pi}{2}} \right) \\ &= \frac{4}{\pi} \frac{\cos(n\pi) - 1}{4n^2} \\ &= \frac{(-1)^n - 1}{\pi n^2} \end{aligned}$$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l} x\right)$$

$$F(t) = \frac{\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(2nt)$$

2.

$$
\begin{aligned}
F(0) = 0 &= \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2} \\
\frac{\pi^2}{4} &= \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} \\
&= \frac{2}{1^2} + 0 + \frac{2}{3^2} + 0 + \frac{2}{5^2} + \dots \\
\frac{\pi^2}{8} &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \\
\frac{\pi^2}{8} &= \sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}
\end{aligned}
$$

Exercise 4

1.

$$
\begin{aligned}
c_n &= \frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(x)e^{\frac{-i2\pi nx}{l}}dx \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{-a} 0 + \int_{-a}^{a} e^{-inx}dx + \int_{a}^{\pi} 0\right) \\
&= \frac{1}{2\pi}\,\frac{e^{-inx}}{-in}\bigg|_{-a}^{a} \\
&= \frac{1}{\pi n}\,\frac{e^{ina}-e^{-ina}}{2i} \\
&= \frac{\sin(na)}{\pi n},\ \ n \neq 0 \\
c_0 &= \frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(x)\,dx \\
&= \frac{1}{2\pi}\int_{-a}^{a} dx \\
&= \frac{a}{\pi}
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{+\infty} c_n e^{\frac{i2\pi nx}{l}} \\
&= \frac{a}{\pi} + \frac{1}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{\sin(na)}{n}e^{inx} \\
&= \frac{a}{\pi} + \frac{1}{\pi}\left(\sum_{n=-\infty}^{-1} \frac{\sin(na)}{n}e^{inx} + \sum_{n=1}^{\infty} \frac{\sin(na)}{n}e^{inx}\right) \\
&= \frac{a}{\pi} + \frac{1}{\pi}\left(\sum_{n=1}^{\infty} \frac{\sin(na)}{n}e^{-inx} + \sum_{n=1}^{\infty} \frac{\sin(na)}{n}e^{inx}\right) \\
&= \frac{a}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(na)}{n}\left(\frac{e^{inx}+e^{-inx}}{2}\right) \\
f(x) &= \frac{a}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(na)\cos(nx)}{n}
\end{aligned}
$$

2.

$$\begin{aligned}
\frac{1}{\pi^2 - e^{3ix}} &= \frac{1}{\pi^2}\frac{1}{1-\dfrac{e^{3ix}}{\pi^2}} \\
&= \frac{1}{\pi^2}\frac{a}{1-r} && a = 1,\ r = \frac{e^{3ix}}{\pi^2} \\
&= \frac{1}{\pi^2}\sum_{n=1}^{\infty} ar^{n-1} && \text{geometric series} \\
&= \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{e^{3ix}}{\pi^2}\right)^{n-1} \\
\frac{1}{\pi^2 - e^{3ix}} &= \frac{1}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{\pi^{2n}}e^{3inx}
\end{aligned}$$

Exercise 5

$$\begin{aligned}
f(-x) &= -x \\
&= -f(x) \\
\implies f(x) &= \text{odd} \\
\implies a_n &= 0\ \forall\ n \in \mathbb{N}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{2}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(x)\sin\left(\frac{2\pi nx}{l}\right)dx \\
&= \frac{2}{2\pi}\int_{-\pi}^{\pi} x\sin\left(\frac{2\pi nx}{2\pi}\right)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx \\
&= \frac{1}{\pi}\left(\frac{-x\cos(nx)}{n}\bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}\right) \\
&= \frac{1}{\pi}\left(\frac{-x\cos(nx)}{n}\bigg|_{-\pi}^{\pi} + \frac{\sin(nx)}{n^2}\bigg|_{-\pi}^{\pi}\right) \\
&= \frac{1}{n\pi}(-2\pi\cos(n\pi) + 0) \\
&= -\frac{2(-1)^n}{n} \\
{b_n}^2 &= \frac{4}{n^2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}}|f(x)|^2\,dx &= \frac{{a_0}^2}{4} + \sum_{n=1}^{\infty}\frac{{a_n}^2 + {b_n}^2}{2} \\
\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx &= 0 + \sum_{n=1}^{\infty}\frac{2}{n^2} \\
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{1}{4\pi}\frac{x^3}{3}\bigg|_{-\pi}^{\pi} \\
\zeta(2) &= \frac{\pi^2}{6}
\end{aligned}$$

Exercise 6

1.

$$
\begin{aligned}
c_n &= \frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(x)e^{\frac{-i2\pi nx}{l}}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\alpha x}e^{\frac{-i2\pi nx}{2\pi}}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(\alpha-in)}dx \\
&= \frac{1}{2\pi}\frac{e^{x(\alpha-in)}}{\alpha-in}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{\pi(\alpha-in)}\frac{e^{\pi(\alpha-in)}-e^{-\pi(\alpha-in)}}{2} \\
&= \frac{\sinh(\pi\alpha-\pi in)}{\pi(\alpha-in)} \\
&= \frac{\alpha+in}{\pi(\alpha^2+n^2)}(\sinh(\pi\alpha)\cosh(-i\pi n)+\cosh(\pi\alpha)\sinh(-i\pi n)) \\
&= \frac{\alpha+in}{\pi(\alpha^2+n^2)}(-i\sin(i\pi\alpha)\cos(\pi n)+\cos(i\pi\alpha)(-i\sin(\pi n))) \\
c_n &= \frac{\sinh(\pi\alpha)(-1)^n(\alpha+in)}{\pi(\alpha^2+n^2)}
\end{aligned}
$$

2.

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{+\infty} c_n e^{\frac{i2\pi nx}{l}} \\
e^{\alpha x} &= \frac{\sinh(\pi\alpha)}{\pi}\sum_{n=-\infty}^{+\infty}\frac{(-1)^n(\alpha+in)}{\alpha^2+n^2}e^{inx}
\end{aligned}
$$

$$
\begin{aligned}
e^0 = \frac{\sinh(\pi\alpha)}{\pi}\sum_{n=-\infty}^{+\infty}\frac{(-1)^n(\alpha+in)}{\alpha^2+n^2} &= 1 \\
\alpha\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{\alpha^2+n^2}+i\sum_{n=-\infty}^{+\infty}\frac{n(-1)^n}{\alpha^2+n^2} &= \frac{\pi}{\sinh(\pi\alpha)} \\
\alpha\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{\alpha^2+n^2}+\left(0+i\sum_{n=1}^{\infty}\left(\frac{n(-1)^n}{\alpha^2+n^2}-\frac{n(-1)^n}{\alpha^2+n^2}\right)\right) &= \frac{\pi}{\sinh(\pi\alpha)} \\
\frac{1}{\alpha^2}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2+n^2} &= \frac{\pi}{\alpha\sinh(\pi\alpha)} \\
\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2+n^2} &= \frac{1}{2}\left(\frac{\pi}{\alpha\sinh(\pi\alpha)}-\frac{1}{\alpha^2}\right)
\end{aligned}
$$