MAU23204: Introduction to Complex Analysis
Midterm Homework due 26/03/2021

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SF Theoretical Physics

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Gaussian Integral

- $$\begin{aligned}
1+e^{-2az_0} &= 0 \\
\implies e^{-2az_0} &= -1 \\
\implies -2az_0 &= (2n+1)(-i\pi), \quad n\in\mathbb{Z} \\
\implies z_0 &= \frac{(2n+1)(i\pi)}{2a}, \quad n\in\mathbb{Z} \\
&= \frac{2n+1}{2}\frac{i\pi}{1+i}\sqrt{\frac{2}{\pi}} \\
&= \left(n+\frac{1}{2}\right)\frac{i\sqrt{2\pi}\,(1-i)}{(1+i)(1-i)} \\
&= \left(n+\frac{1}{2}\right)\frac{\sqrt{2\pi}\,(1+i)}{2} \\
z_0 &= \left(n+\frac{1}{2}\right)(1+i)\sqrt{\frac{\pi}{2}} \\
&= a\left(n+\frac{1}{2}\right), \quad n\in\mathbb{Z}
\end{aligned}$$

$$\begin{aligned}
g(z) &= \frac{1}{f(z)} \\
&= \frac{1+e^{-2az}}{e^{-z^2}} \\
g(z_0) &= \frac{1-1}{e^{-z^2}} \\
&= 0 \\
g'(z_0) &= \frac{-2a\,e^{-2az_0}}{e^{-z_0^2}} + \left(1+e^{-2az_0}\right)\frac{d}{dz}\left(\frac{1}{e^{-z^2}}\right)\bigg|_{z_0} \\
&= -2a\,\frac{e^{-2az_0}}{e^{-z_0^2}} + 0
\end{aligned}$$

$$\begin{aligned}
a^2 &= \frac{\pi}{2}(1+i)^2 \\
&= \frac{\pi}{2}\,2i \\
&= i\pi
\end{aligned}$$

$$\begin{aligned}
-2a\,z_0 &= -2a^2\left(n+\frac{1}{2}\right) \\
&= -i\pi\,(2n+1)
\end{aligned}
\qquad\qquad
\begin{aligned}
-z_0^2 &= -a^2\left(n+\frac{1}{2}\right)^2 \\
&= -\frac{i\pi}{4}\,(2n+1)^2
\end{aligned}$$

$(2n+1)^2 = 1, 9, 25, 49, 81, \ldots = 4(0)+1, 4(2)+1, 4(6)+1, 4(12)+1, 4(20)+1, \ldots$, i.e. any odd number squared is always 1 plus some multiple of 4. Thus we can rewrite $-z_0^2 = \left(-\frac{i\pi}{4} - q\,i\pi\right)$, for some $q \in \mathbb{N}$.

$$\begin{aligned}
g'(z_0) &= -2a\,\frac{e^{-i\pi(2n+1)}}{e^{-\frac{i\pi}{4}}\,e^{-q\,i\pi}} \\
&= \frac{-2a(-1)}{e^{-\frac{i\pi}{4}}} \\
&= \frac{2a}{\frac{1-i}{\sqrt{2}}} \\
&= 2\sqrt{2}\,\frac{(1+i)(1+i)}{(1-i)(1+i)}\sqrt{\frac{\pi}{2}} \\
&= 2\sqrt{\pi}\,\frac{2i}{2} \\
&= 2i\sqrt{\pi} \\
&\neq 0
\end{aligned}$$

Thus $\frac{1}{g} = f$ has a pole of order $m = 1$ at z_0.

$$\begin{aligned}
\mathrm{Res}_{z_0} f &= \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right) \\
&= \lim_{z\to z_0} (z-z_0)\,\frac{e^{-z^2}}{1+e^{-2az}} \\
&= e^{-z_0^2} \lim_{z\to z_0} \frac{z-z_0}{1+e^{-2az}} \\
&= e^{-z_0^2} \lim_{z\to z_0} \frac{1}{-2a\,e^{-2az}} && \text{by L'Hôpital's Rule} \\
&= \frac{e^{-z_0^2}}{-2a\,e^{-2az_0}} \\
&= \frac{1}{g'(z_0)} \\
&= \frac{1}{2i\sqrt{\pi}} \\
\mathrm{Res}_{z_0} f &= -\frac{i}{2\sqrt{\pi}}
\end{aligned}$$

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\begin{aligned}
f(z)-f(z+a) &= \frac{e^{-z^2}}{1+e^{-2az}} - \frac{e^{-(z+a)^2}}{1+e^{-2a(z+a)}} \\
&= \frac{e^{-z^2}}{1+e^{-2az}} - \frac{e^{-z^2-2az-a^2}}{1+e^{-2az-2a^2}} \\
&= e^{-z^2}\left(\frac{1}{1+e^{-2az}} - \frac{e^{-2az-a^2}}{1+e^{-2az-2a^2}}\right) \\
&= e^{-z^2}\left(\frac{1\left(1+e^{-2az-2a^2}\right)-e^{-2az-a^2}\left(1+e^{-2az}\right)}{\left(1+e^{-2az}\right)\left(1+e^{-2az-2a^2}\right)}\right) \\
&= e^{-z^2}\left(\frac{1+e^{-2az-2a^2}-e^{-2az-a^2}-e^{-4az-a^2}}{1+e^{-2az-2a^2}+e^{-2az}+e^{-4az-2a^2}}\right) \\
&= e^{-z^2}\left(\frac{1+\left(e^{-a^2}\right)^2 e^{-2az}-e^{-a^2}e^{-2az}-e^{-a^2}e^{-4az}}{1+\left(e^{-a^2}\right)^2 e^{-2az}+e^{-2az}+\left(e^{-a^2}\right)^2 e^{-4az}}\right) \\
a^2 = i\pi \implies e^{i\pi} = -1 \implies & e^{-z^2}\left(\frac{1+e^{-2az}+e^{-2az}+e^{-4az}}{1+e^{-2az}+e^{-2az}+e^{-4az}}\right) \\
&= e^{-z^2}
\end{aligned}
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\begin{aligned}
z_0 \text{ inside rectangle} &\implies -R < \operatorname{Re}(z_0) < R \text{ and } 0 < \operatorname{Im}(z_0) < \operatorname{Im}(a) \\
&\implies -R < \left(n+\frac{1}{2}\right)\sqrt{\frac{\pi}{2}} < R \text{ and } 0 < \left(n+\frac{1}{2}\right)\sqrt{\frac{\pi}{2}} < \sqrt{\frac{\pi}{2}} \\
&\implies |R| > \left(n+\frac{1}{2}\right)\sqrt{\frac{\pi}{2}} \text{ and } -\frac{1}{2} < n < \frac{1}{2} \\
&\implies |R| > \left(n+\frac{1}{2}\right)\sqrt{\frac{\pi}{2}} \text{ and } n = 0 \\
&\implies |R| > \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ and } z_0 = \frac{a}{2}
\end{aligned}
$$

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\begin{aligned}
\int_\gamma f &= 2i\pi \sum_k W(\gamma, z_k) \operatorname{Res}_{z_k} f \\
&= \begin{cases} 2i\pi\,(1)\,\dfrac{-i}{2\sqrt{\pi}} \text{ if } |R| > \dfrac{1}{2}\sqrt{\dfrac{\pi}{2}} \\ 0 \text{ if } |R| \leq \dfrac{1}{2}\sqrt{\dfrac{\pi}{2}} \end{cases}
\end{aligned}
$$

$$
\int_\gamma f = \begin{cases} \sqrt{\pi} \text{ if } |R| > \dfrac{1}{2}\sqrt{\dfrac{\pi}{2}} \\ 0 \text{ otherwise} \end{cases}
$$

- $$\lim_{R\to\infty}\int_{\gamma_{2,4}} f = \lim_{R\to\infty}\left(\pm\int_0^{\sqrt{\frac{\pi}{2}}} \frac{e^{-(\pm R+it)^2}}{1+e^{-2a(\pm R+it)}}\, i\, dt\right) \qquad \text{as } z = R+it \implies \frac{dz}{dt} = i$$

$$\left|e^{-(\pm R+it)^2}\right| = e^{-\left(R^2-t^2\right)} \le e^{-\left(R^2-\frac{\pi}{2}\right)}\ \forall t\in\left[0,\sqrt{\frac{\pi}{2}}\right]$$
$$\text{If } \left|1+e^{-2a(\pm R+it)}\right| = 0 \implies \pm R+it = \frac{1}{2}\sqrt{\frac{\pi}{2}}+\frac{i}{2}\sqrt{\frac{\pi}{2}}$$
$$\text{But } |R| > \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ and } R\to\infty \implies \left|1+e^{-2a(\pm R+it)}\right| \ge \epsilon \text{ for some } \epsilon > 0$$

$$\begin{aligned}
\lim_{R\to\infty}\left|\int_{\gamma_{2,4}} f\right| &\le \lim_{R\to\infty} M\, L_\gamma \\
&\le \lim_{R\to\infty} \frac{e^{-\left(R^2-\frac{\pi}{2}\right)^2}}{\epsilon}\sqrt{\frac{\pi}{2}} \\
&\le 0 \\
\implies \lim_{R\to\infty}\int_{\gamma_{2,4}} f = 0
\end{aligned}$$

- $$\begin{aligned}
\lim_{R\to\infty}\left(\int_{\gamma_1} f+\int_{\gamma_3} f\right) &= \lim_{R\to\infty}\left(\int_{-R}^{R} f(x)\,dx + \int_{R}^{-R} f\left(x+i\sqrt{\frac{\pi}{2}}\right)dx\right) \\
&= \lim_{R\to\infty}\left(\int_{-R}^{R} f(x)\,dx - \int_{-R}^{R} f\left(x+i\sqrt{\frac{\pi}{2}}\right)dx\right) \\
&= \lim_{R\to\infty}\left(\int_{-R}^{R} f(x)\,dx - \int_{-R-\sqrt{\frac{\pi}{2}}}^{R-\sqrt{\frac{\pi}{2}}} f\left(y+\sqrt{\frac{\pi}{2}}+i\sqrt{\frac{\pi}{2}}\right)dy\right) \\
&= \int_{-\infty}^{\infty} f(x)\,dx - \int_{-\infty}^{\infty} f(y+a)\,dy \\
&= \int_{-\infty}^{\infty} \left(f(x)-f(x+a)\right)dx \\
&= \int_{-\infty}^{\infty} e^{-x^2}\,dx
\end{aligned}$$

- $$\begin{aligned}
\sqrt{\pi} &= \int_\gamma f \\
&= \int_{\gamma_1} f+\int_{\gamma_2} f+\int_{\gamma_3} f+\int_{\gamma_4} f \\
&= \left(\int_{\gamma_2} f+\int_{\gamma_4} f\right)+\left(\int_{\gamma_1} f+\int_{\gamma_3} f\right) \\
&= 0+\int_{-\infty}^{\infty} e^{-x^2}\,dx \\
\implies \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}
\end{aligned}$$

Harmonic functions

- $$\begin{aligned}
\nabla^2(u\cdot v) &= (u\cdot v)_{xx} + (u\cdot v)_{yy} \\
&= (u_x\cdot v + u\cdot v_x)_x + (u_y\cdot v + u\cdot v_y)_y \\
u_x = v_y,\ u_y = -v_x \implies & (u_x\cdot v - u\cdot u_y)_x + (u_y\cdot v + u\cdot u_x)_y \\
&= u_{xx}\cdot v + u_x\cdot v_x - u_x\cdot u_y - u\cdot u_{yx} + u_{yy}\cdot v + u_y\cdot v_y + u_y\cdot u_x + u\cdot u_{xy} \\
u_x = v_y,\ u_y = -v_x \implies & v\cdot u_{xx} - u_x\cdot u_y - u_x\cdot u_y - u\cdot u_{yx} + v\cdot u_{yy} + u_y\cdot u_x + u_y\cdot u_x + u\cdot u_{xy} \\
&= v\cdot(u_{xx}+u_{yy}) + 2(-u_x\cdot u_y + u_y\cdot u_x) + u\cdot(-u_{yx}+u_{xy})
\end{aligned}$$

  Since u is harmonic, it solves the Laplace equation, i.e. $\nabla^2 u = u_{xx} + u_{yy} = 0$, and so the first term is 0. The second term is clearly 0. Since $u_{xy} = u_{yx}$, the third term is also 0. Thus we have $\nabla^2(u\cdot v) = 0$, and thus $u\cdot v$ is harmonic.

- $$\begin{aligned}
u(x_0,y_0) &= \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\,d\theta && \text{(mean value property)} \\
\implies u(0,0) &= \frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\,d\theta \\
\implies |u(0,0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\,d\theta\right| \\
&\le \frac{1}{2\pi}\int_0^{2\pi} |u(r\cos\theta, r\sin\theta)|\,d\theta && \text{(estimation lemma)} \\
&\le \frac{1}{2\pi}\int_0^{2\pi} |u(0,0)|\,d\theta && (u(x,y)\le u(0,0)\ \forall (x,y)\in \text{unit disc}) \\
&= |u(0,0)| && (u(0,0)\ \text{constant})
\end{aligned}$$

  $$\begin{aligned}
|u(0,0)| \le \frac{1}{2\pi}\int_0^{2\pi} &|u(r\cos\theta, r\sin\theta)|\,d\theta \le |u(0,0)| \\
\implies \frac{1}{2\pi}\int_0^{2\pi} &|u(r\cos\theta, r\sin\theta)|\,d\theta = |u(0,0)|
\end{aligned}$$

  This is true for any $0 < r \le 1$, and so the average of $|u|$ on any circle of radius less than 1 centred at (0, 0) is the same as the absolute value of the maximum. Since the function is continuous, the value of u cannot instantaneously switch between positive and negative $u(0,0)$, and so we have

  $$\frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\,d\theta = u(0,0),$$

  i.e. the average about any circle smaller than the unit disc centred at (0,0) is the same as the maximum. This can only be true if u is a constant on the unit disc.

Laurent series

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\begin{aligned}
\frac{1}{(z-1)(z-3)} &= \frac{\alpha}{z-1} + \frac{\beta}{z-3} \\
\implies \alpha(z-3) + \beta(z-1) &= 1 \\
\implies -2\alpha = 1 \text{ and } 2\beta &= 1 \text{ by setting } z = 1 \text{ and } z = 3 \\
\implies f(z) &= \frac{1}{2}\left(-\frac{1}{z-1} + \frac{1}{z-3}\right)
\end{aligned}
$$

$$
\begin{aligned}
|z| < 1 \implies -\frac{1}{z-1} &= \frac{1}{1-z} \\
&= \sum_{n=0}^{\infty} z^n \\
|z| > 1 \implies -\frac{1}{z-1} &= -\frac{1}{z}\frac{1}{1-\frac{1}{z}} \\
&= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \\
&= -\sum_{n=1}^{\infty} z^{-n}
\end{aligned}
$$

$$
\begin{aligned}
|z| < 3 \implies \frac{1}{z-3} &= -\frac{1}{3}\frac{1}{1-\frac{z}{3}} \\
&= -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n \\
|z| > 3 \implies \frac{1}{z-3} &= \frac{1}{z}\frac{1}{1-\frac{3}{z}} \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n \\
&= \frac{1}{3}\sum_{n=1}^{\infty} 3^n z^{-n}
\end{aligned}
$$

$$
f(z) = \begin{cases}
\frac{1}{2}\left(\sum_{n=0}^{\infty} z^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right) & 0 \le |z| < 1 \\
\frac{1}{2}\left(-\sum_{n=1}^{\infty} z^{-}n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right) & 1 < |z| < 3 \\
\frac{1}{2}\left(-\sum_{n=1}^{\infty} z^{-n} + \frac{1}{3}\sum_{n=1}^{\infty} 3^n z^{-n}\right) & 3 < |z| < \infty
\end{cases}
$$

$$
f(z) = \begin{cases}
\frac{1}{2}\sum_{n=0}^{\infty} z^n\left(1 - \frac{1}{3^{n+1}}\right) & 0 \le |z| < 1 \\
-\frac{1}{2}\left(\frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{1}{z^n} + \frac{1}{3}\left(\frac{z}{3}\right)^n\right)\right) & 1 < |z| < 3 \\
\frac{1}{2}\sum_{n=1}^{\infty}\frac{3^{n-1}-1}{z^n} & 3 < |z| < \infty
\end{cases}
$$