PYU22T20: Chaos and Complexity
Assignment due 23/03/2021

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I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at
http://www.tcd.ie/calendar.
I have completed the Online Tutorial in avoiding plagiarism 'Ready, Steady, Write', located at http://tcd-ie.libguides.com/plagiarism/ready-steady-write.

Question 1

$$\begin{aligned}
\dot{x}|_{(x_0,y_0,z_0)} &= p(y_0 - x_0) \\
&= 0 \\
\implies x_0 &= y_0 \\
\dot{y}|_{(x_0,y_0,z_0)} &= -x_0 z_0 + r x_0 - y_0 \\
&= x_0(r - z_0 - 1) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
x_0 &= 0 \\
\implies y_0 &= 0 \\
\dot{z}|_{(x_0,y_0,z_0)} &= x_0 y_0 - b z_0 \\
&= -b z_0 \\
&= 0 \\
\implies z_0 &= 0
\end{aligned}$$

$$\begin{aligned}
x_0 &\neq 0 \\
\implies z_0 &= r - 1 \\
\dot{z}|_{(x_0,y_0,z_0)} &= x_0 y_0 - b z_0 \\
&= x_0^2 + b - br \\
&= 0 \\
\implies x_0 &= \pm\sqrt{b(r-1)} \\
\implies y_0 &= \pm\sqrt{b(r-1)} \\
x_0, y_0 \neq 0 \implies r &> 1
\end{aligned}$$

The fixed points are thus $(0,0,0)$ and $\left(\pm\sqrt{b(r-1)},\, \pm\sqrt{b(r-1)},\, r-1\right)$, if $r > 1$.

$$\begin{aligned}
J &= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} \\
&= \begin{pmatrix} -p & p & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}
\end{aligned}$$

$$J_1 = \begin{pmatrix} -p & p & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \qquad J_2 = \begin{pmatrix} -p & p & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{pmatrix}$$

Question 2

$$
\begin{aligned}
J_1 &= \begin{pmatrix} -p & p & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \\
\det(J_1 - \lambda I) &= \begin{vmatrix} & -p-\lambda & p & 0 \\ (& r & -1-\lambda & 0 \\ & 0 & 0 & -b-\lambda \end{vmatrix} \\
&= (-p-\lambda)(-1-\lambda)(-b-\lambda) - p(r(-b-\lambda)) \\
&= (-b-\lambda)\left(p + \lambda(p+1) + \lambda^2 - pr\right) \\
&= 0
\end{aligned}
$$

$$\lambda_0 = -b \qquad\qquad \lambda_\pm = \frac{-p-1 \pm \sqrt{(p+1)^2 + 4p(r-1)}}{2}$$

The real part of λ_0 and λ_- is negative for any value of p, r and b. A fixed point is only stable if the real parts of all the eigenvalues are negative, and so for $(0,0,0)$ to be stable, the real part of λ_+ must be negative. Thus, either the term in the square root of λ_+ is less than or equal to 0, or it is positive and λ_+ is real and negative.

Thus for $(0,0,0)$ to be stable, either

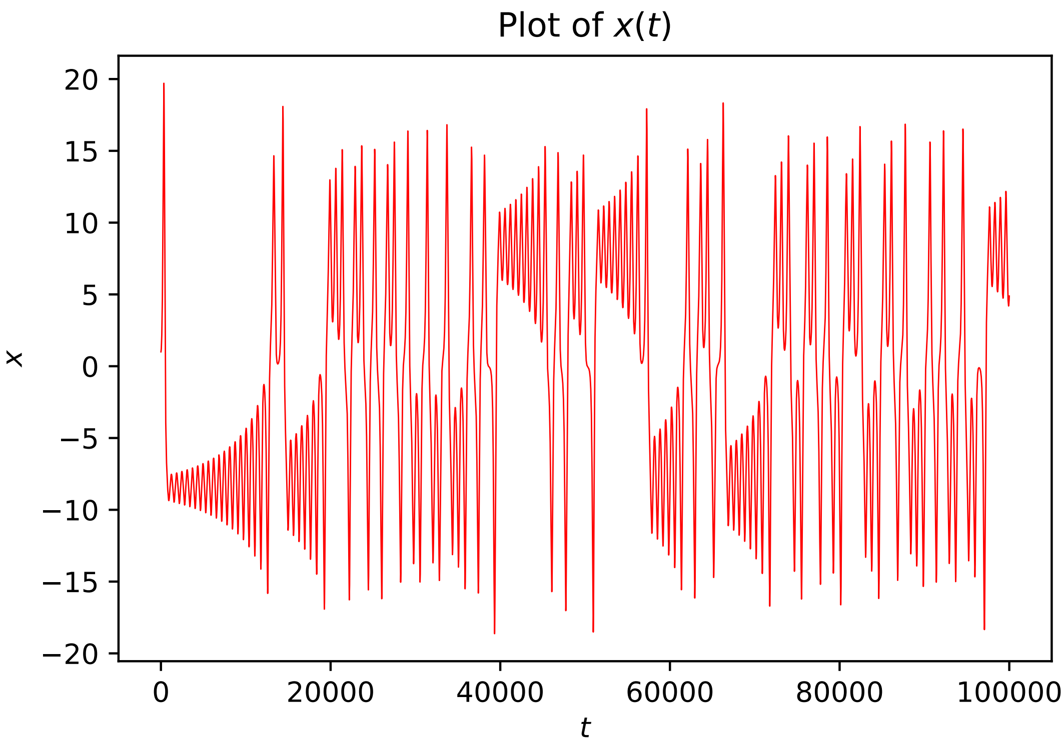
$$(p+1)^2 + 4p(r-1) \leq 0$$

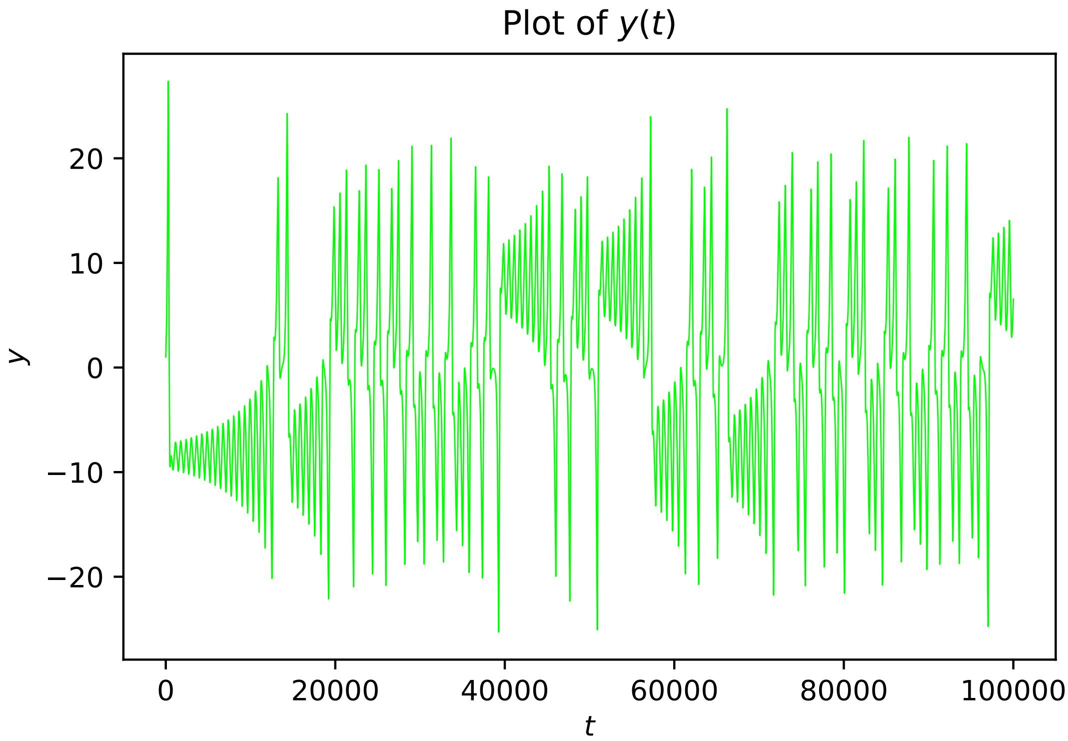
or

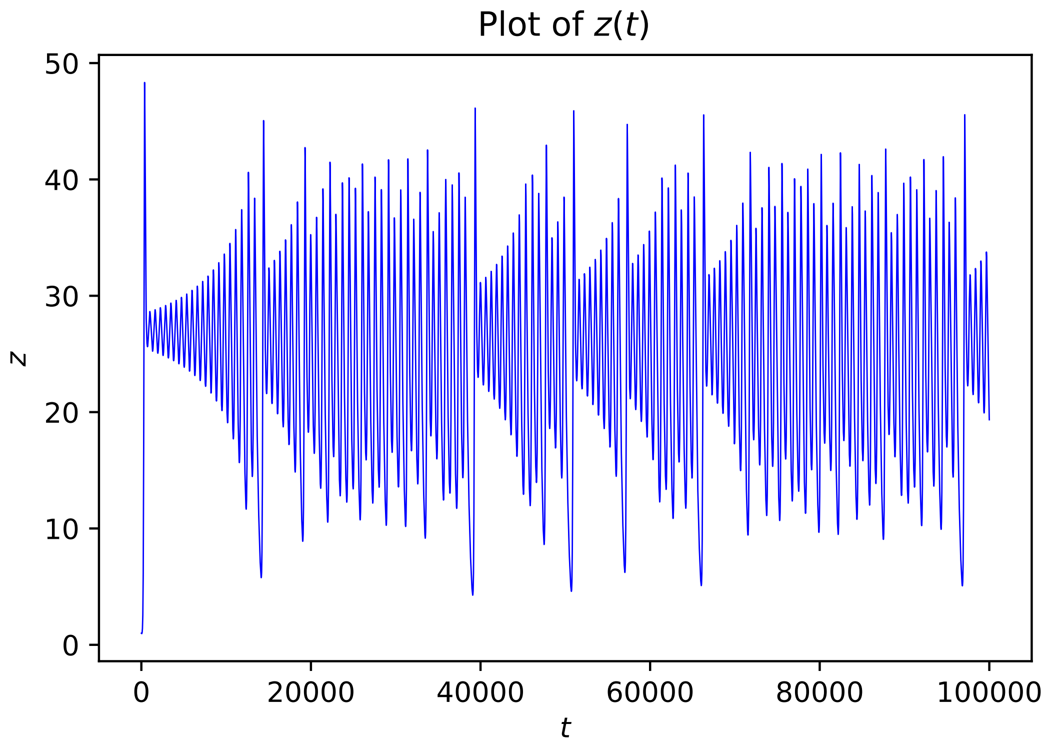
$$(p+1)^2 + 4p(r-1) > 0 \text{ and } -p-1+\sqrt{(p+1)^2 + 4p(r-1)} < 0.$$

Question 3

The following graphs of $x(t)$, $y(t)$, and $z(t)$ were plotted.

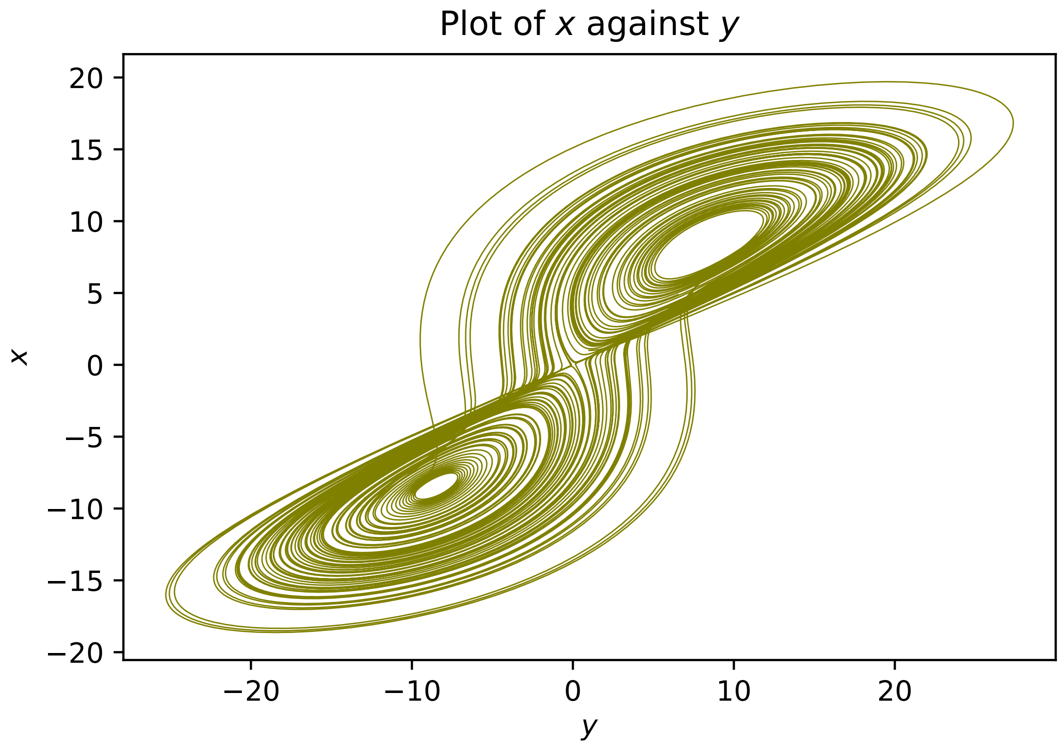


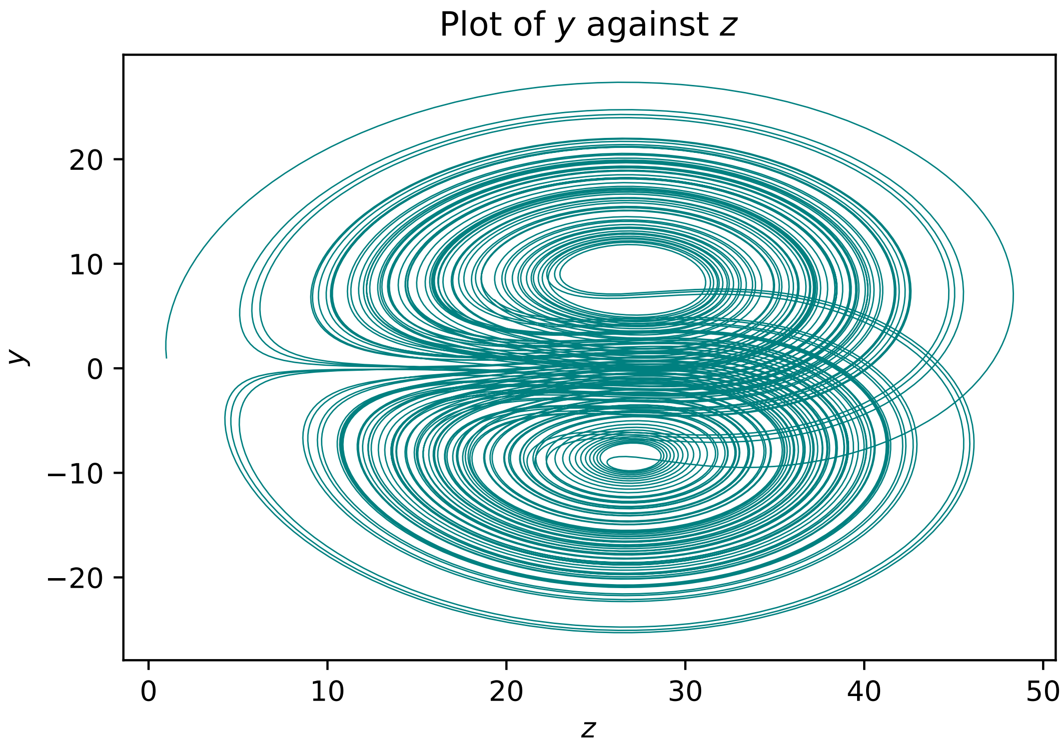


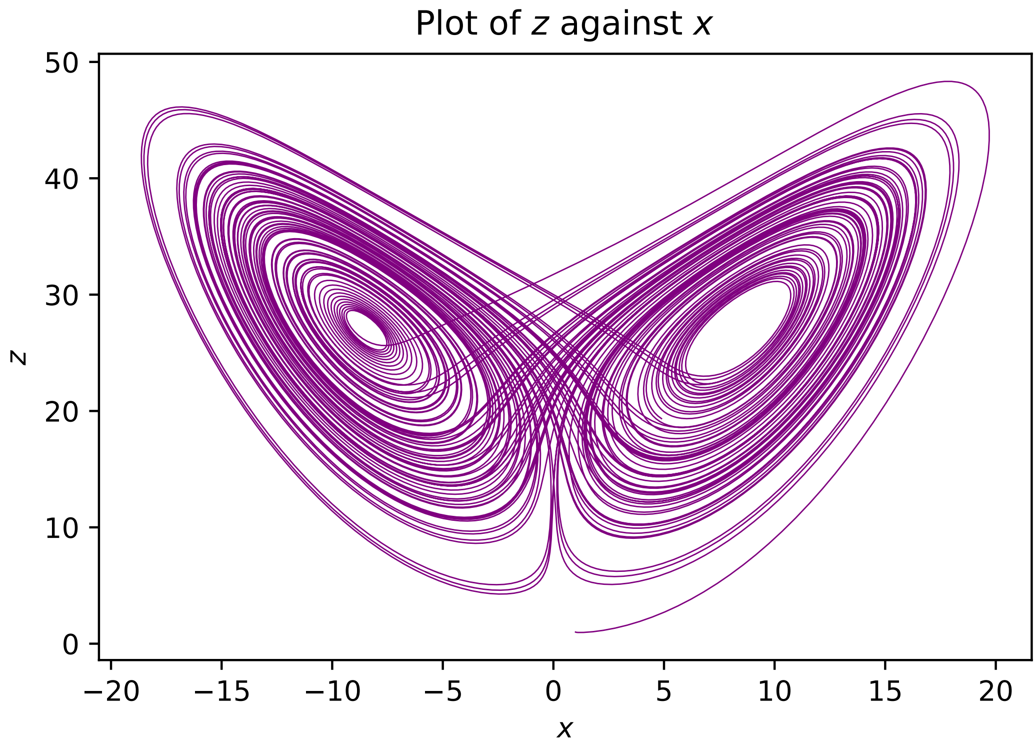


The plots of $x(t)$ and $y(t)$ are almost identical, the main difference being the amplitude of their motion. It is clear, however, that none of the three motions seem to be periodic, which suggests that the motion is chaotic.

The following graphs of x vs y, y vs z and z vs x were plotted.

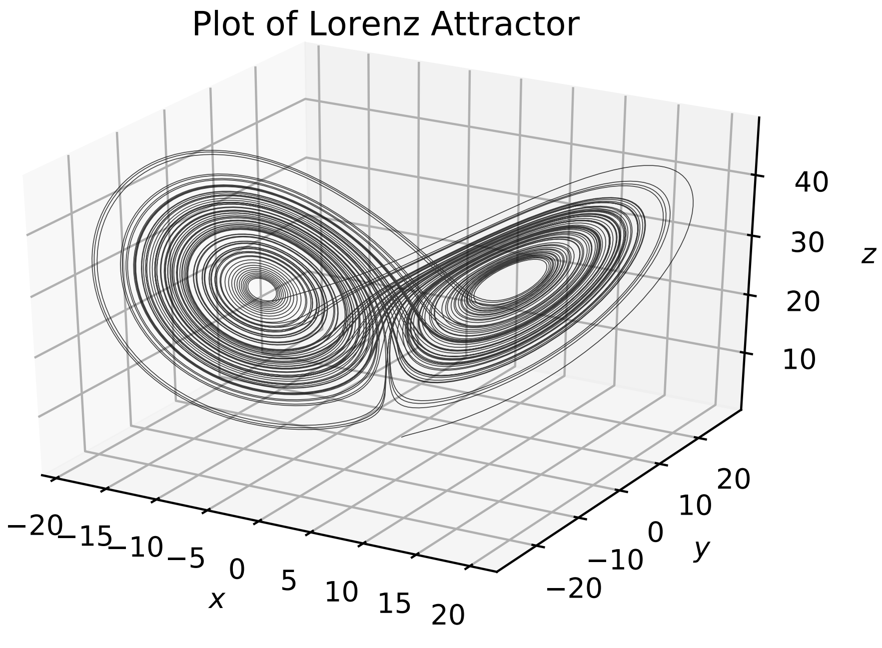






Although it is hard to make out the starting point in the plot of x against y, it is clear from the other two plots that the motion never reaches the starting point, which would also suggest chaotic motion.

The following graph of x, y and z was plotted in a 3D plot. (I have also submitted a GIF of the plot showing various angles)



This plot makes it obvious that the motion is chaotic, as it spirals around two points, but never comes back to the starting point of $(1, 1, 1)$.