MAU11204: Analysis on the Real Line
Homework 9 due 14/03/2021

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Problem 1

Define a function $g_A : \mathbb{R} \to \{0, 1\}$ such that $g(x) = 1$ if $x \in A$, and $g(x) = 0$ if $x \notin A$, i.e.

$$g_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Since $[a, b] = [a, c) \cup [c, b]$, we know that $g_{[a,b]} = g_{[a,c)} + g_{[c,b]}$. We also know that, since f maps from $[a, b]$, $f(x) = f(x)g_{[a,b]}(x)$. We can thus say that

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_{-\infty}^{\infty} f(x)g_{[a,b]}\,dx \\
&= \int_{-\infty}^{\infty} f(x)\left(g_{[a,c)}(x) + g_{[c,b]}(x)\right)dx \\
&= \int_{-\infty}^{\infty} f(x)g_{[a,c)}(x)\,dx + \int_{-\infty}^{\infty} f(x)g_{[c,b]}(x)\,dx \\
&= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx
\end{aligned}$$

Since f is integrable on $[a, b]$, then $\int_a^b f(x)\,dx$ exists. This means that both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ must also exist, and so f must be integrable on $[a, c]$ and $[c, b]$.

Problem 2

Since the function f is greater than or equal to 0 for all points in $[a, b]$, there is no partition P such that $L(f, P) < 0$, and thus $L(f, P) \geqslant 0$ for all partitions. This means that $L(f) = \sup\{L(f, P)\} \geqslant 0$. Now consider $U(f, P)$. We have that, for any ϵ, there is some P such that $U(f, P) \leqslant \epsilon$. We can thus make ϵ as small as we want, and there will always be some partition P such that $U(f, P)$ is less than or equal to ϵ. If the set $\{U(f, P)\}$ had an infimum $\delta > 0$, then if we consider some $\epsilon < \delta$, we would have an element of this set that is less than the infimum, which is a contradiction. The infimum of this set must be less than or equal to 0. Since the function is always greater than 0, there is no upper sum less than 0, and so $U(f) = \inf\{U(f, P)\} = 0$. Since for any partition P we have that $L(f, P) \leqslant U(f, P)$, we also have that $L(f) \leqslant U(f)$. We thus have that $0 \leqslant L(f) \leqslant U(f) = 0$, and so $L(f) = U(f) = 0$. Therefore, f is integrable, and its integral is 0.