MAU11204: Analysis on the Real Line
Homework 7 due 31/03/2021

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Problem 1

Label $A_x = A \cap (x - \epsilon_x, x + \epsilon_x)$, where ϵ_x is some $\epsilon > 0$ such that $A \cap (x - \epsilon, x + \epsilon)$ is countable, for some x. Consider the sets A_{x_i}, for all $x_i \in A$. For each of these sets, consider $p_i, q_i \in \mathbb{Q}$ such that $x_i - \epsilon_{x_i} \leqslant p_i \leqslant x_i \leqslant q_i \leqslant x_i + \epsilon_{x_i}$. If A_{x_i} contains $x_a \in A$ such that $x_a < x_i$, then restrict p_i so that $x_a < p_i \leqslant x_i$. Likewise, if A_{x_i} contains x_b such that $x_i < x_b$, then restrict q_i so that $x_i \leqslant q_i < x_b$. These restrictions are reasonable, as each A_{x_i} is countable, and so there must exist a rational number in between any two pair of elements in A_{x_i}.

Now consider the map $\psi : A \to \mathbb{Q} \times \mathbb{Q}$ given by $x_k \mapsto (p_k, q_k)$. Since each pair (p_i, q_i) is defined such that the only element of A in this interval can be x_i, then each pair (p_i, q_i) in the image of ψ either corresponds to no elements of A or exactly one element of A. Thus, the map ψ must be injective, and so $|A| \leqslant |\mathbb{Q} \times \mathbb{Q}| = |\mathbb{N}|$, i.e. A must be at most countable.

Problem 2

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, then $\lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n}} = \sqrt{x^2} = |x|$, i.e. $f_n(x)$ converges to $|x|$. Say that $f_n(x)$ does not uniformly converge to $|x|$. Thus, for some x, we have

$$
\begin{aligned}
& \exists \epsilon > 0 \text{ s.t. } \nexists N : n \geqslant N \implies |f_n(x) - f(x)| < \epsilon \\
& \implies \exists \epsilon > 0 \text{ s.t. } \forall N : n \geqslant N \implies \left| \sqrt{x^2 + \frac{1}{n}} - |x| \right| \geqslant \epsilon \\
\sqrt{x^2 + \frac{1}{n}} > |x| \geqslant 0 \ & \implies \exists \epsilon > 0 \text{ s.t. } \forall n : \sqrt{x^2 + \frac{1}{n}} \geqslant \epsilon + |x| \\
& \implies \exists \epsilon > 0 \text{ s.t. } \forall n : x^2 + \frac{1}{n} \geqslant \epsilon^2 + x^2 + 2\epsilon|x| \\
& \implies \exists \epsilon > 0 \text{ s.t. } \forall n : \frac{1}{n} \geqslant \epsilon\left(\epsilon + 2|x|\right) \\
& \implies \exists \epsilon > 0 \text{ s.t. } \forall n : \frac{1}{2}\left(\frac{1}{n\epsilon} - \epsilon\right) \geqslant |x| \\
\text{Pick } n > \frac{1}{\epsilon^2} \implies \frac{1}{n\epsilon} < \epsilon \ & \implies \exists \epsilon > 0 \text{ s.t. } \exists n : 0 > \frac{1}{2}\left(\frac{1}{n\epsilon} - \epsilon\right) \geqslant |x|
\end{aligned}
$$

This is a contradiction, as we have that $|x| < 0$ for some x. Thus, $f_n(x)$ must uniformly converge to $|x|$.

The derivative of the function f_n is $f_n'(x) = \frac{1}{2}\left(x^2 + \frac{1}{n}\right)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}} \to \frac{x}{|x|} = f'(x)$. If $f_n'(x)$ converges uniformly, then we must have that for any $\epsilon > 0$ there is an N such that $n \geqslant N$ implies that $|f_n'(x) - f'(x)| < \epsilon$, for all x. If we let $x = 0$, then we have that $f'(x)$ is undefined. Thus, there is no ϵ such that $|f_n'(0) - f'(0)| < \epsilon$, for any n, and so $f_n'(x)$ does not uniformly converge.

Problem 3

Consider the function $f_A : \mathbb{R} \to \mathbb{R}$ given by $f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$, for some $A \subseteq \mathbb{R}$. The set of these functions is a subset of all functions $f : \mathbb{R} \to \mathbb{R}$, and so $|\{f_{A_i}\}| \leqslant \left|\mathbb{R}^{\mathbb{R}}\right|$, for any $\{A_i\}$. Since each f_A is uniquely determined by the set A, we have that $|\{f_{A_i}\}| = |\{A_i\}|$. Thus, if we prove that the cardinality of the set of all possible subsets of $\mathbb{R}$ is larger than the cardinality of $\mathbb{R}$ itself, we can show that $\mathbb{R}$ and $\mathbb{R}^{\mathbb{R}}$ do not have the same cardinality, i.e. $|\mathbb{R}| < |\{A_i\}| = |\{f_{A_i}\}| \leqslant \left|\mathbb{R}^{\mathbb{R}}\right|$.

Suppose there exists a surjection $\alpha : \mathbb{R} \to \{A_i\}$, where $\{A_i\}$ is the set of all subsets of $\mathbb{R}$. Let Γ be the set of all real numbers that are not contained in their own image in α, i.e. $\Gamma = \{x \in \mathbb{R} \mid x \notin \alpha(x)\} \subseteq \mathbb{R}$. Since α is surjective and Γ is a subset of $\mathbb{R}$, then there must be some $y \in \mathbb{R}$ such that $\alpha(y) = \Gamma$. If $y \in \Gamma$, then $y \in \alpha(y) = \Gamma$. If $y \notin \Gamma$, then $y \in \alpha(y) = \Gamma$. Thus we have $y \in \Gamma \iff y \notin \Gamma$, which is a contradiction. Thus there is no surjection that maps from $\mathbb{R}$ to $\{A_i\}$. Now consider the map $\beta : \mathbb{R} \to \{A_i\}$ given by $\beta(x) = \{x\}$. This mapping is clearly injective, as if $\beta(x) = \beta(y)$ then $\{x\} = \{y\}$, and so $x = y$. Thus there exists an injection from $\mathbb{R}$ to $\{A_i\}$, but no surjection, and so we have that $|\mathbb{R}| < \{A_i\}$.

Therefore $\mathbb{R}$ and $\mathbb{R}^{\mathbb{R}}$ do not have the same cardinality.