MAU11204: Analysis on the Real Line
Homework 6 due 24/03/2021

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Problem 1

Consider the set $A = (-1, 0] \cup (1, 2)$ and the function $f : A \to \mathbb{R}$ given by $f(x) = \begin{cases} x \text{ if } x \in (-1,0] \\ x-1 \text{ if } x \in (1,2) \end{cases}$.

The infimum and supremum of A are -1 and 2 respectively, and so A is bounded.

The functions $\alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \to \mathbb{R}$ given by $\alpha(x) = x$ and $\beta(x) = x - 1$ are clearly continuous. Thus, the restrictions $\alpha_{(-1,0]} : (-1, 0] \to \mathbb{R}$ and $\beta_{(1,2)} : (1, 2) \to \mathbb{R}$ must also be continuous. However, α restricted to map from $(-1, 0]$ is the same as f restricted to map from $(-1, 0]$, and β restricted to map from $(1, 2)$ is the same as f restricted to map from $(1, 2)$. Thus f is continuous on $(-1, 0] \cup (1, 2)$, which is simply A, and so f is continuous.

$\alpha_{(-1,0]}$ and $\beta_{(1,2)}$ are both clearly injective, and share no common values in their images (as $\alpha_{(-1,0]}((-1, 0]) \cap \beta_{(1,2)}((1, 2)) = (-1, 0] \cap (0, 1) = \varnothing$), and so f is injective.

Consider the sequence $x_n = \frac{1}{2^{n+1}}$ for $n \in \mathbb{N}$. This sequence converges to 0 as n tends to infinity, however is always larger than 0. Since $x_n \in \left(0, \frac{1}{2}\right] \subset (0, 1) = f((1, 2))$, and $x_n \to 0$, we have that $f^{-1}(x_n) \to 1$. However, $f^{-1}(0) = 0 \neq 1$. Thus there exists a convergent sequence $x_n \to x$ such that $f^{-1}(x_n)$ does not converge to x, and so f^{-1} cannot be continuous.

Problem 2

Say that f is not injective, i.e. there exist $x_a \neq x_b$ such that $f(x_a) = f(x_b)$. Since $[x_a, x_b] \subseteq A$ is bounded and closed in A, it is compact. By the extreme value theorem, $f|_{[x_a,x_b]} : [x_a, x_b] \to \mathbb{R}$ attains a maximum and a minimum at some points. If $f(x_a) = f(x_b)$ is both a maximum and a minimum, then the function on $[x_a, x_b]$ must be constant. If this was the case, then every point on $[x_a, x_b]$ would attain the same value, and so the function on this interval would not be injective. Thus, for the property to be satisfied, there must be a minimum or maximum attained on (x_a, x_b). Say that this minimum or maximum is attained at $q \in (x_a, x_b)$. For any $x \in (x_a, x_b)\backslash\{q\}$, we have that $f|_{[x_a,x_b]}(x) > f|_{[x_a,x_b]}(q)$ if q is a minimum point, and $f|_{[x_a,x_b]}(x) < f|_{[x_a,x_b]}(q)$ if q is a maximum point. Since f is continuous, it must be defined for all $x \in A$, and so we have that for any $\epsilon > 0$ there exist $x_1, x_2 \in (x_a, x_b)$ where $q - \epsilon \leq x_1 < q < x_2 \leq q + \epsilon$ such that $f(x_1) = f(x_2)$, i.e. there is no $\epsilon > 0$ such that $f|_{(x-\epsilon,x+\epsilon)} : (q - \epsilon, q + \epsilon) \to \mathbb{R}$ is injective. The property is therefore not satisfied if f is not injective. Thus, if the property is satisfied, then f must be injective.