MAU11204: Analysis on the Real Line
Homework 5 due 10/03/2021

Ruaidhrí Campion
19333850
SF Theoretical Physics

I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at http://www.tcd.ie/calendar.
I have completed the Online Tutorial in avoiding plagiarism 'Ready, Steady, Write', located at http://tcd-ie.libguides.com/plagiarism/ready-steady-write.

Problem 1

$$\begin{aligned}
f\left(\frac{a_i+b_i}{2}\right) > 0 &\implies I_{i+1} = \left[a_i, \frac{a_i+b_i}{2}\right] \subset [a_i, b_i] = I_i \\
f\left(\frac{a_i+b_i}{2}\right) \not> 0 &\implies I_{i+1} = \left[\frac{a_i+b_i}{2}, b_i\right] \subset [a_i, b_i] = I_i \\
&\implies I_{i+1} \subset I_i \ \forall i \in \mathbb{N} \\
&\implies I_{i+1} \cap I_i = I_{i+1} \ \forall i \in \mathbb{N}
\end{aligned}$$

$$\begin{aligned}
\bigcap_{i=0}^{\infty} I_i &= \lim_{n\to\infty} \bigcap_{i=0}^{n} I_i \\
&= \lim_{n\to\infty} I_n \qquad \text{(from above)} \\
&= \lim_{n\to\infty} [a_n, b_n]
\end{aligned}$$

Thus it is sufficient to show that $\lim_{n\to\infty} [a_n, b_n] = \{x_0\}$.

Bolzano's theorem states that for any interval $[a, b]$ such that $f(a) \leq 0$ and $f(b) \geq 0$ for a continuous function f, there must be a root in the interval. There thus exists a root on the interval $I_0 = [a, b]$. Say that for some interval $I_i = [a_i, b_i]$, we have $f(a_i) \leq 0$ and $f(b_i) \geq 0$. Say that $f\left(\frac{a_i+b_i}{2}\right) > 0$. Then $I_{i+1} = [a_{i+1}, b_{i+1}] = \left[a_i, \frac{a_i+b_i}{2}\right]$, and we have $f(a_{i+1}) = f(a_i) \leq 0$ and $f(b_{i+1}) = f\left(\frac{a_i+b_i}{2}\right) > 0$. Say that $f\left(\frac{a_i+b_i}{2}\right) \leq 0$. Then $I_{i+1} = [a_{i+1}, b_{i+1}] = \left[\frac{a_i+b_i}{2}, b_i\right]$, and we have $f(a_{i+1}) = f\left(\frac{a_i+b_i}{2}\right) \leq 0$ and $f(b_{i+1}) = f(b_i) \geq 0$. In both of these situations, $f(a_{i+1}) \leq 0$ and $f(b_{i+1}) \geq 0$, and so by Bolzano's theorem, I_{i+1} has a root, if $f(a_i) \leq 0$ and $f(b_i) \geq 0$. Thus, by induction, I_i contains a root for any $i \in \mathbb{N}$.

Since $a_i \leq b_i \ \forall i \in \mathbb{N}$, then $a_i \leq \frac{a_i+b_i}{2} \ \forall i \in \mathbb{N}$. After each iteration we have either $a_{i+1} = a_i$ or $a_{i+1} = \frac{a_i+b_i}{2}$, and so a_i either remains constant or increases after each iteration. Since there is always a root in the interval $[a_i, b_i] \ \forall i \in \mathbb{N}$, then $a_i \leq x_0 \ \forall i \in \mathbb{N}$ for some x_0 such that $f(x_0) = 0$. Since a_i can only increase, and is always less than some x_0, it must converge to a point $x_1 \leq x_0$. We can similarly show that, since $b_i \geq \frac{a_i+b_i}{2}$, b_i can only decrease, and since it must also be greater than some x_0, it converges to a point $x_2 \geq x_0$.

If $x_1 \neq x_0$ or $x_2 \neq x_0$, then $\lim_{n\to\infty} [a_n, b_n] = [x_1, x_2]$, which is an interval, as $x_1 \neq x_2$. If $x_1 = x_2 = x_0$, then $\lim_{n\to\infty} [a_n, b_n] = [x_0, x_0] = \{x_0\}$. Thus we only need to show that this method cannot converge to an interval.

Say that $\lim_{n\to\infty} I_n$ results in some interval. This would imply that we can still perform an infinite number of iterations on this interval. Thus, if this method converges to an interval, it has not converged, which is a contradiction. Thus this method cannot converge to an interval.

Therefore $\lim_{n\to\infty} [a_n, b_n] = \{x_0\}$, and so $\bigcap_{i=0}^{\infty} I_i = \{x_0\}$.

Problem 2

Let $(x_i)_{i\in\mathbb{N}^*}$ be bounded. Let $(x_{i_j})_{j\in\mathbb{N}^*}$ be a subsequence of $(x_i)_{i\in\mathbb{N}^*}$. Since $(x_i)_{i\in\mathbb{N}^*}$ is bounded, and $\{x_{i_j}\} \subseteq \{x_i\}$, then $(x_{i_j})_{j\in\mathbb{N}^*}$ must also be bounded. From the Bolzano-Weierstrass theorem, which states that every bounded sequence has a convergent subsequence, $(x_{i_j})_{j\in\mathbb{N}^*}$ must have a convergent subsequence. Thus, if $(x_i)_{i\in\mathbb{N}^*}$ is bounded, then every subsequence of $(x_i)_{i\in\mathbb{N}^*}$ has a convergent (sub)subsequence.

Let $(x_i)_{i\in\mathbb{N}^*}$ be unbounded. If $(x_i)_{i\in\mathbb{N}^*}$ is unbounded from above, then we can choose a subsequence $(x_{i_j})_{j\in\mathbb{N}^*}$ that is always increasing and does not converge, i.e. $x_{i_{n+1}} - x_{i_n} \geq \varepsilon \ \forall n \in \mathbb{N}^*$ for some $\varepsilon > 0$. Any subsequence of $(x_{i_j})_{j\in\mathbb{N}^*}$ will also clearly not converge, as it will always be increasing, and the difference in subsequent terms must be greater than or equal to ε. Thus, not every subsequence of $(x_i)_{i\in\mathbb{N}^*}$ has a convergent (sub)subsequence. A similar argument can be made if $(x_i)_{i\in\mathbb{N}^*}$ is unbounded from below, where we can choose a subsequence that is always decreasing and does not converge, and any subsequence of this subsequence will also not converge. Thus, if $(x_i)_{i\in\mathbb{N}^*}$ is unbounded, not every subsequence of $(x_i)_{i\in\mathbb{N}^*}$ has a convergent (sub)subsequence. By contraposition, if every subsequence of $(x_i)_{i\in\mathbb{N}^*}$ has a convergent (sub)subsequence, then $(x_i)_{i\in\mathbb{N}^*}$ msut be bounded.

Therefore $(x_i)_{i\in\mathbb{N}^*}$ is bounded $\iff$ every subsequence of $(x_i)_{i\in\mathbb{N}^*}$ has a convergent (sub)subsequence.

Problem 3

Since f is uniformly continuous, then for all $\alpha > 0$ there exists a $\beta > 0$ such that $|x_n - x_m| < \beta \implies |f(x_n) - f(x_m)| < \alpha$ for all $x_m, x_n \in A$. Since $(x_i)_{i\in\mathbb{N}^*}$ is a Cauchy sequence, then for all $\beta > 0$ there exists an N such that $|x_n - x_m| < \beta$ for all $m, n \geq N$. Combining these leads to the following: for all $\alpha > 0$ there exists an N such that $|f(x_n) - f(x_m)| < \alpha$ for all $m, n \geq N$. By definition, this implies that $(f(x_i)_{i\in\mathbb{N}^*})$ is also a Cauchy sequence.