MAU11204: Analysis on the Real Line
Homework 4 due 03/03/2021

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Problem 1

In order to show that f is not continuous, it must first be shown that $\mathbb{R}$ and $\varnothing$ are the only sets that are both open and closed in $\mathbb{R}$.

Let $V \subset \mathbb{R}$ be an open and closed set that is non-empty and is not $\mathbb{R}$ itself. Thus there exists $x_a, x_b \in \mathbb{R}$ such that $x_a \in V$ and $x_b \notin V$. Say that $x_a < x_b$. $[x_a, x_b] \cap V$ is a finite intersection of closed sets, and so is closed. Thus, $x_b > x_c = \sup([x_a, x_b] \cap V) \in V$. Since V is also open, then there exists an $\varepsilon > 0$ such that $(x_c - \varepsilon, x_c + \varepsilon) \subseteq V$, i.e. such that $x_c + \varepsilon \in V$. This leads to a contradiction, as there then exists an element in V that is larger than the supremum of V intersected with another closed set. A similar proof follows if $x_a > x_b$, where the infimum is considered instead of the supremum. Thus, without a loss of generality, this is a sufficient proof that there are no sets both open and closed in $\mathbb{R}$ other than $\mathbb{R}$ and $\varnothing$.

$$
\begin{aligned}
f \text{ is continuous} &\implies f^{-1}(\alpha) \text{ is open iff } \alpha \text{ is open } \wedge f^{-1}(\beta) \text{ is closed iff } \beta \text{ is closed} \\
f^{-1}((0.9, 1.1)) = f^{-1}([0.9, 1.1]) = A &\implies A \text{ is both open and closed} \\
&\implies A = \mathbb{R} \vee A = \varnothing \\
A \subset \mathbb{R} \wedge (A = \mathbb{R} \vee A = \varnothing) &\implies \text{contradicton} \\
&\therefore f \text{ is not continuous}
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
\text{Assume } \bigcup_{i\in I} A_i \text{ is not connected} &\implies \exists x < y < z : x, z \in \bigcup_{i\in I} A_i \wedge y \notin \bigcup_{i\in I} A_i \\
&\implies \exists \alpha, \beta \in I : x_a \notin A_\beta \wedge x_b \notin A_\alpha \ \forall x_a \in A_\alpha \ \forall x_b \in A_\beta \\
&\implies \exists \alpha, \beta \in I : A_\alpha \cap A_\beta = \varnothing \\
&\implies \bigcap_{i\in I} A_i = \varnothing \\
\therefore \bigcap_{i\in I} A_i \neq \varnothing &\implies \bigcup_{i\in I} A_i \text{ is connected (by contraposition)}
\end{aligned}
$$

Problem 3

Say a continuous function $f : A \to \mathbb{R}$ does not have the intermediate value property, i.e. there exists a z in between $f(a)$ and $f(b)$ for some $a, b \in A$ such that there is no c in between a and b where $f(c) = z$. Thus there exists $\mathbb{R} \ni z \notin f(A)$ such that $f(a) < z < f(b)$ for some $f(a), f(b) \in f(A)$, $f(a) < f(b)$. Thus by definition, $f(A)$ is not connected. $f(A)$ is therefore contained in $\alpha \cup \beta$, where $\alpha, \beta \subset \mathbb{R}$ are disjoint, non-empty and open in $\mathbb{R}$, and $f(A)$ has at least one element in each of α and β.

$f^{-1}(\alpha)$ and $f^{-1}(\beta)$ are

- disjoint (as if $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ shared an element c, then $f(c)$ would be shared by α and β, which are disjoint),
- non-empty (as both α and β contain at least one element of $f(A)$, and so $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ must contain at least one element of A), and
- open (as the inverse map of an open set is open for a continuous function).

Since $f(A)$ is contained in $\alpha \cup \beta$, then $f^{-1}(\alpha) \cup f^{-1}(\beta) = A$. Thus, if f does not have the intermediate value property, then A can be written as a union of disjoint, non-empty open sets, and so is not connected. Therefore, if A is connected, then any continuous $f : A \to \mathbb{R}$ must have the intermediate value property.

Let $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = x$. If we let $f = g|_A$ such that $f : A \to \mathbb{R}$, then this function is continuous, as if g is continuous for all $x \in \mathbb{R}$, then g is continuous for all $x \in A \subset \mathbb{R}$. Assume A is not connected. Thus there exists a point $\mathbb{R} \ni c \notin A$ such that $a < b < c$ for some $a, b \in A$, $a < b$. Thus there exists a point $z \in \mathbb{R}$ in between $f(a) = a$ and $f(b) = b$ such that there is no point in A that attains z, i.e. $f^{-1}(\{z\}) = \varnothing$, and so f does not have the intermediate value property. This means that if A is not connected, there exists a continuous function $f : A \to \mathbb{R}$ that does not have the intermediate value property. Thus, if every continuous function $f : A \to \mathbb{R}$ has the intermediate value property, then A must be connected.

Therefore $A \subset \mathbb{R}$ is connected $\iff$ every continuous function $f : A \to \mathbb{R}$ has the intermediate value property.

Problem 4

(a)

Since we are asked to prove that the image of any bounded subset of $\mathbb{R}$ is bounded, we have to consider any bounded $B \subseteq \mathbb{R}$. Since B is bounded, then $B \neq \mathbb{R}$. If $B = \varnothing$ then, for any function $f : \mathbb{R} \to \mathbb{R}$, the image of B will simply be $\varnothing$. Since $\varnothing$ is bounded, then the image of this bounded set is indeed bounded. Thus we only need to show that the image of any proper bounded subset of $\mathbb{R}$ is bounded for a continuous function, i.e. we need only consider a bounded $B \subset \mathbb{R}$.

Let's say that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the image of a bounded $B \subset \mathbb{R}$ is not bounded. This means there exists a finite limit point x_0 of B such that $\lim_{x \to x_0} f(x) = \pm\infty$. If x_0 is a finite limit point of B, and B is a subset of $\mathbb{R}$, then $x_0 \in \mathbb{R}$. If $\lim_{x \to x_0} f(x) = \pm\infty$, then either $f(x_0)$ is not defined, or $f(x_0) = y_0$ for some finite y_0. This means that $f(x_0) \neq \lim_{x \to x_0} f(x)$, and so f is not continuous at x_0. However, if f is continuous, then it must be continuous for all $x \in \mathbb{R}$. This is a contradiction, and so there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the image of a bounded $B \subset \mathbb{R}$ is not bounded.

Therefore, for a continuous function $f : \mathbb{R} \to \mathbb{R}$, the image of any bounded subset of $\mathbb{R}$ is bounded.

(b)

Consider the function $f(x) = \frac{1}{x-y}$, for some y. This function is continuous at all points $x \neq y$, and discontinuous at $x = y$. Thus this function is continuous for any set $A \subset \mathbb{R}$ that does not contain y. Say $A = \mathbb{R} \backslash \{y\}$. If we say that $A \supset B = (y, b)$ for some $b > y$, then the image of this bounded set is $\left(\frac{1}{b-y}, \infty\right)$, which is not bounded. Thus not all images of bounded subsets of A are bounded. If we were to restrict A to be some other set where y is a limit point, we run into the same problem, as we can still form a bounded $B \subset A$ such that y is a limit point of B, which results in an unbounded image of a bounded set, even though f is continuous on A. If we were to restrict A such that y was not a limit point, but it had some other finite limit point $z \notin A$, then we run into the same problem when we consider the function $f(x) = \frac{1}{x-z}$, which is continuous on A.

If we again considered $f(x) = \frac{1}{x-y}$, but now considered an A that contains y, then this function is not continuous on A, and so it does not need to be considered further. If this set had a finite limit point $z \notin A$, then we run into the previous problem of the image of a bounded subset not being bounded, even if the function is continuous. If $z \in A$, this problem is fixed. This can be repeated for all finite limit points of A. If we make A contain all its finite limit points, this simply makes A closed. Thus any continuous function on A evaluated at a finite limit point must be defined, and so the image of any bounded subset of A must therefore also be bounded.

Thus, for the image of a bounded subset of A to also be bounded for all continuous functions on A, A must be closed.