MAU11204: Analysis on the Real Line
Homework 3 due 24/02/2021

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I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at
http://www.tcd.ie/calendar.
I have completed the Online Tutorial in avoiding plagiarism 'Ready, Steady, Write', located at http://tcd-ie.libguides.com/plagiarism/ready-steady-write.

Problem 1

$$\begin{aligned}
\text{Assume } \overline{\mathbb{R}\backslash\mathbb{Q}} = A \neq \mathbb{R} &\implies \exists \text{ closed set } A \subset \mathbb{R} : \mathbb{R}\backslash\mathbb{Q} \subseteq A \\
&\implies \exists \text{ open set } A^c \subset \mathbb{R} : A^c \subseteq \mathbb{Q} \\
&\implies \exists \epsilon > 0 : (x-\epsilon, x+\epsilon) \subseteq A^c \subseteq \mathbb{Q}\ \forall x \in A^c \\
\text{For a given } x : \text{if } \epsilon \text{ is rational} &\implies \left[x - \frac{\epsilon}{\sqrt{2}}, x + \frac{\epsilon}{\sqrt{2}}\right] \subset (x-\epsilon, x+\epsilon) \subseteq A^c \subseteq \mathbb{Q} \\
&\implies x + \frac{\epsilon}{\sqrt{2}} \in \mathbb{Q} \\
x \in A^c \subseteq \mathbb{Q} \implies x \text{ is rational} \land \epsilon \text{ is rational} &\implies x + \frac{\epsilon}{\sqrt{2}} \text{ is irrational} \\
x + \frac{\epsilon}{\sqrt{2}} \in \mathbb{Q} \land x + \frac{\epsilon}{\sqrt{2}} \text{ is irrational} &\implies \text{contradiction} \\
&\implies \epsilon \text{ cannot be rational} \\
\text{if } \epsilon \text{ is irrational} &\implies \left[x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right] \subset (x-\epsilon, x+\epsilon) \subseteq A^c \subseteq \mathbb{Q} \\
&\implies x + \frac{\epsilon}{2} \in \mathbb{Q} \\
x \in A^c \subseteq \mathbb{Q} \implies x \text{ is rational} \land \epsilon \text{ is irrational} &\implies x + \frac{\epsilon}{2} \text{ is irrational} \\
x + \frac{\epsilon}{2} \in \mathbb{Q} \land x + \frac{\epsilon}{2} \text{ is irrational} &\implies \text{contradiction} \\
&\implies \epsilon \text{ cannot be irrational} \\
\epsilon \text{ is neither rational nor irrational} &\implies \text{contradiction} \\
&\implies A = \mathbb{R} \\
&\therefore \overline{\mathbb{R}\backslash\mathbb{Q}} = \mathbb{R}
\end{aligned}$$

Problem 2

$$\begin{aligned}
\inf\{x_n\} \leq x_n \leq \sup\{x_n\}\ \forall n \in \mathbb{N} &\implies x_n \in [\inf\{x_n\}, \sup\{x_n\}]\ \forall n \in \mathbb{N} \\
x_n \to x &\implies \inf\{x_n\} \leq x \leq \sup\{x_n\} \\
&\implies x \in [\inf\{x_n\}, \sup\{x_n\}]
\end{aligned}$$

$$\begin{aligned}
x_n \in (0,1)\ \forall n \in \mathbb{N} &\implies 0 \leq \inf\{x_n\} \wedge \sup\{x_n\} \leq 1 \\
\text{Assume } \inf\{x_n\} = 0 &\implies x = 0 \vee \exists x_n = 0 \\
(x = 0 \vee \exists x_n = 0) \wedge (x \in (0,1) \wedge x_n \in (0,1)) &\implies \text{contradiction} \\
&\implies \inf\{x_n\} > 0 \\
\text{Assume } \sup\{x_n\} = 1 &\implies x = 1 \vee \exists x_n = 1 \\
(x = 1 \vee \exists x_n = 1) \wedge (x \in (0,1) \wedge x_n \in (0,1)) &\implies \text{contradiction} \\
&\implies \sup\{x_n\} < 1 \\
\inf\{x_n\} > 0 \wedge \sup\{x_n\} < 1 &\implies [\inf\{x_n\}, \sup\{x_n\}] \subset (0,1)
\end{aligned}$$

$$\therefore \exists a, b \in \mathbb{R} : x_n \in [a,b] \wedge x \in [a,b] \wedge [a,b] \subset (0,1) \qquad (\text{e.g. } a = \inf\{x_n\},\ b = \sup\{x_n\})$$

Problem 3

Let $A = \mathbb{Q}$ and $B = \mathbb{R}\backslash\mathbb{Q}$

$$\begin{aligned}
A \cap B &= \mathbb{Q} \cap (\mathbb{R}\backslash\mathbb{Q}) \\
&= \emptyset \\
\overline{A \cap B} &= \overline{\emptyset} \\
&= \emptyset \text{ as } \emptyset \text{ is closed}
\end{aligned}
\qquad
\begin{aligned}
\overline{A} &= \overline{\mathbb{Q}} \\
&= \mathbb{R} \\
\overline{A} \cap \overline{B} &= \mathbb{R} \cap \mathbb{R} \\
&= \mathbb{R}
\end{aligned}
\qquad
\begin{aligned}
\overline{B} &= \overline{\mathbb{R}\backslash\mathbb{Q}} \\
&= \mathbb{R}
\end{aligned}$$

$$\overline{A \cap B} = \emptyset \neq \mathbb{R} = \overline{A} \cap \overline{B}$$

$$\therefore \overline{A \cap B} \neq \overline{A} \cap \overline{B}$$

Problem 4

(a)

$$\begin{aligned}
\exists A_i : f|_{A_i} \text{ is not continuous at } x &\implies \exists \text{ nbhd } U \text{ of } f(x) \in B : \nexists \text{ nbhd } V \text{ of } x \in A_i : f(V) \subseteq U \\
x \in A_i \subseteq A &\implies x \in A \\
&\implies \exists \text{ nbhd } U \text{ of } f(x) \in B : \nexists \text{ nbhd } V \text{ of } x \in A : f(V) \subseteq U \\
&\implies f \text{ is not continuous at } x \\
&\implies f \text{ is not continuous} \\
f \text{ is continuous} &\implies \text{each } f|_{A_i} \text{ is continuous (by contraposition)}
\end{aligned}$$

$$\begin{aligned}
f|_{A_i} \text{ is continuous at } x &\implies \forall \text{ nbhds } U \text{ of } f(x) \in B\ \exists \text{ nbhd } V \text{ of } x \in A_i : f(V) \subseteq U \\
f|_{A_i} \text{ is continuous} &\implies f|_{A_i} \text{ is continuous at } x\ \forall x \in A_i \\
&\implies \forall \text{ nbhds } U \text{ of } f(x) \in B\ \exists \text{ nbhd } V \text{ of } x \in A_i : f(V) \subseteq U\ \forall x \in A_i \\
\text{Each } f|_{A_i} \text{ is continuous} &\implies f|_{A_i} \text{ is continuous at } x\ \forall x \in A_i\ \forall A_i \subseteq A \\
&\implies \forall \text{ nbhds } U \text{ of } f(x) \in B\ \exists \text{ nbhd } V \text{ of } x \in A_i : f(V) \subseteq U\ \forall x \in A_i\ \forall A_i \subseteq A \\
&\implies \forall \text{ nbhds } U \text{ of } f(x) \in B\ \exists \text{ nbhd } V \text{ of } x \in A : f(V) \subseteq U\ \forall x \in A \\
&\implies f \text{ is continuous at } x\ \forall x \in A \\
&\implies f \text{ is continuous}
\end{aligned}$$

$$\therefore f \text{ is continuous} \iff \text{each } f|_{A_i} \text{ is continuous}$$

(b)

The proofs above do not assume if A is a finite or an infinite union of subsets A_i, and all assumptions made apply for both a finite and infinite union of subsets. Thus, the same is true if we allow for an infinite number of A_i's.