MAU11204: Analysis on the Real Line
Homework 2 due 17/02/2021

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Problem 1

$$\begin{aligned}
\text{Assume } a < \sup\{r \in \mathbb{Q} \mid r < a\} &\implies a \in \{r \in \mathbb{Q} \mid r < a\} \\
a \notin \{r \in \mathbb{Q} \mid r < a\} \land a \in \{r \in \mathbb{Q} \mid r < a\} &\implies \text{contradiction} \\
&\therefore a \not< \sup\{r \in \mathbb{Q} \mid r < a\}
\end{aligned}$$

$$\begin{aligned}
\text{Assume } a > \sup\{r \in \mathbb{Q} \mid r < a\} &\implies \exists q \in \mathbb{Q} : \sup\{r \in \mathbb{Q} \mid r < a\} < q < a \\
q \in \mathbb{Q} \land q < a &\implies q \in \{r \in \mathbb{Q} \mid r < a\} \\
\sup\{r \in \mathbb{Q} \mid r < a\} < q \in \{r \in \mathbb{Q} \mid r < a\} &\implies \text{contradiction} \\
&\therefore a \not> \sup\{r \in \mathbb{Q} \mid r < a\}
\end{aligned}$$

Therefore $a = \sup\{r \in \mathbb{Q} \mid r < a\}$.

Problem 2

Since every finite intersection of open subsets is open, then if two subsets of the real numbers A and B are open, their intersection will also be open. If we define $A = (a, \infty)$ and $B = (-\infty, b)$, with $a < b$, then their intersection is equal to (a, b). If (a, ∞) and $(-\infty, b)$ are open subsets, then their intersection (a, b) is open.

$$\begin{aligned}
\text{Say } (a, \infty) \text{ is not open} &\implies \exists x \in (a, \infty) : (x - \varepsilon, x + \varepsilon) \nsubseteq (a, \infty)\ \forall \varepsilon \in \mathbb{R} \\
(x + \varepsilon \in (a, \infty) \forall x \in (a, \infty), \varepsilon \in \mathbb{R}) &\implies \exists x \in (a, \infty) : (x - \varepsilon, x) \nsubseteq (a, \infty)\ \forall \varepsilon \in \mathbb{R} \\
\text{Say } x > a &\implies \exists \varepsilon \in \mathbb{R} : x - \varepsilon > a \implies \exists \varepsilon \in \mathbb{R} : (x - \varepsilon, x) \subseteq (a, \infty) \\
&\implies x \leq a \\
&\implies x \notin (a, \infty)
\end{aligned}$$

This is a contradiction, as (a, ∞) being not open implies that there exists an element in (a, ∞) that is also not an element. Thus, (a, ∞) must be an open subset. A similar argument can be made for $(-\infty, b)$, where if it is assumed that it is not open, there is an element that is not an element. Thus $(-\infty, b)$ must also be open.

Since (a, ∞) and $(-\infty, b)$ are open subsets, their intersection (a, b) must also be open.

Problem 3

$$\begin{aligned}
\text{Say that } \sup A \text{ exists} &\implies \sup A = \min\{x \in F \mid x \text{ is an upper bound}\} \\
&\implies -\sup A = -\min\{x \in F \mid x \text{ is an upper bound}\} \\
&\implies -\sup A = \max\{-x \in F \mid x \text{ is an upper bound}\} \\
&\implies -\sup A = \max\{-x \in F \mid -x \text{ is a lower bound}\} \\
&\implies -\sup A = \inf(-A)
\end{aligned}$$

$$\begin{aligned}
\text{Say that } \inf B \text{ exists} &\implies -\inf B = -\max\{x \in F \mid x \text{ is a lower bound}\} \\
&\implies -\inf B = \min\{-x \in F \mid x \text{ is a lower bound}\} \\
&\implies -\inf B = \min\{-x \in F \mid -x \text{ is an upper bound}\} \\
&\implies -\inf B = \sup(-B) \\
&\implies \inf B = -\sup(-B) \\
\inf(-A) \text{ exists} &\implies \inf(-A) = -\sup A
\end{aligned}$$

$$\begin{aligned}
-\sup A = \inf(-A) &\implies \sup A \text{ exists} \wedge \inf(-A) \text{ exists} \\
\sup A \text{ exists} &\implies -\sup A = \inf(-A) \\
\inf(-A) \text{ exists} &\implies -\sup A = \inf(-A)
\end{aligned}$$

$$\therefore \sup A \text{ exists} \iff \inf(-A) \text{ exists} \iff -\sup A = \inf(-A)$$

Problem 4

$$\begin{aligned}
\text{Assume } \max A > \sup A &\implies \exists a \in A : a > \sup A \\
&\implies \text{contradiction} \\
&\therefore \max A \not> \sup A
\end{aligned}$$

$$\begin{aligned}
\text{Assume } \max A < \sup A &\implies \exists q \in \mathbb{Q} : \max A < q < \sup A \\
&\implies q \text{ is an upper bound} \\
q \text{ is an upper bound} \wedge q < \sup A &\implies \text{contradiction} \\
&\therefore \max A \not< \sup A
\end{aligned}$$

Therefore if $\max A$ exists, $\max A = \sup A$.