MAU11204: Analysis on the Real Line
Homework 1 due 10/02/2021

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Problem 1

1.

$A \iff B$:

$A \downarrow B \rightarrow$	T	F
T	T	F
F	F	T

$A \implies B$:

$A \downarrow B \rightarrow$	T	F
T	T	F
F	T	T

2.

$\neg(A \lor B)$: if A or B are true, then $A \lor B$ will be true, and so $\neg(A \lor B)$ will be false. If both A and B are false, then $A \lor B$ will be false, and so $\neg(A \lor B)$ will be true. Thus the truth table for $\neg(A \lor B)$ is

$A \downarrow B \rightarrow$	T	F
T	F	F
F	F	T

$(\neg A) \land (\neg B)$: if A is true, then $\neg A$ will be false, and so $(\neg A) \land (\neg B)$ will be false. If B is true, then $\neg B$ will be false, and so $(\neg A) \land (\neg B)$ will be false. If both A and B are false, then both $\neg A$ and $\neg B$ will be true, and so $(\neg A) \land (\neg B)$ will be true. Thus the truth table for $(\neg A) \land (\neg B)$ is

$A \downarrow B \rightarrow$	T	F
T	F	F
F	F	T

These have the same truth tables, and so $\neg(A \lor B) \iff (\neg A) \land (\neg B)$

3.

If we label $A = (\forall m \in \mathbb{N}\ \forall n \in \mathbb{N} : m < n)$ and $B = (\exists x \in X : m < x \land x < n)$, we can construct a truth table for the statement $\forall m \in \mathbb{N}\ \forall n \in \mathbb{N} : m < n \implies (\exists x \in X : m < x \land x < n)$ from part 1.

$A \downarrow B \rightarrow$	T	F
T	T	F
F	T	T

The negation to this statement corresponds to the following truth table

$A \downarrow B \rightarrow$	T	F
T	F	T
F	F	F

This is the truth table for $A \land \neg B$, and so the negation to the statement is

$$(\forall m \in \mathbb{N}\ \forall n \in \mathbb{N} : m < n) \land \neg(\exists x \in X : m < x \land x < n)$$

which is equivalent to

$$(\forall m \in \mathbb{N}\ \forall n \in \mathbb{N} : m < n) \land (\nexists x \in X : m < x \land x < n)$$

Problem 2

1.

$$\begin{aligned}
\text{Choose some } x \in X_1 &\implies f(\{x\}) \subseteq f(X_1) \\
&\implies x \in f^{-1}(f(\{x\})) \subseteq f^{-1}(f(X_1)) \\
\big(x \in X_1 \implies x \in f^{-1}(f(X_1))\big) &\implies X_1 \subseteq f^{-1}(f(X_1))
\end{aligned}$$

$$\begin{aligned}
\text{Assume } f \text{ is injective. Choose some } x \in f^{-1}(f(X_1)) &\implies f(x) \in f(X_1) \\
&\implies \exists x' \in X_1 : f(x) = f(x') \\
(f \text{ is injective}) \wedge f(x) = f(x') &\implies x = x' \in X_1 \\
\big(x \in f^{-1}(f(X_1)) \implies x \in X_1\big) &\implies f^{-1}(f(X_1)) \subseteq X_1 \\
f^{-1}(f(X_1)) \subseteq X_1 \wedge X_1 \subseteq f^{-1}(f(X_1)) &\implies X_1 = f^{-1}(f(X_1)) \\
\therefore f \text{ is injective} &\implies X_1 = f^{-1}(f(X_1))\ \forall X_1 \subset X
\end{aligned}$$

$$\begin{aligned}
\text{Assume } f \text{ is not injective} &\implies \exists x, x' \in X : f(x) = f(x') \wedge x \neq x' \\
&\implies \exists x, x' \in X : \{x, x'\} \subseteq f^{-1}(f(\{x\})) \\
&\implies \exists x \in X : \{x\} \neq f^{-1}(f(\{x\})) \\
&\implies \exists X_1 \subset X : X_1 \neq f^{-1}(f(X_1)) \\
&\implies \neg\big(X_1 = f^{-1}(f(X_1))\ \forall X_1 \subset X\big) \\
\therefore f^{-1}(f(X_1)) = X_1\ \forall X_1 \subset X &\implies f \text{ is injective (by contraposition)}
\end{aligned}$$

Thus f is injective $\iff X_1 = f^{-1}(f(X_1))\ \forall X_1 \subset X$

2.

$$\begin{aligned}
\text{Choose some } y \in f(f^{-1}(Y_1)) &\implies \exists x \in f^{-1}(Y_1) : y = f(x) \\
&\implies \exists x \in X : f(x) \in Y_1 \wedge y = f(x) \\
&\implies y \in Y_1 \\
\big(y \in f(f^{-1}(Y_1)) \implies y \in Y_1\big) &\implies f(f^{-1}(Y_1)) \subseteq Y_1
\end{aligned}$$

$$\begin{aligned}
\text{Assume } f \text{ is surjective. Choose some } y \in Y_1 &\implies \exists x \in X : y = f(x) \wedge y \in Y_1 \\
&\implies \exists x \in X : f(x) \in Y_1 \wedge y = f(x) \\
&\implies \exists x \in f^{-1}(Y_1) : y = f(x) \\
&\implies y \in f(f^{-1}(Y_1)) \\
\big(y \in Y_1 \implies y \in f(f^{-1}(Y_1))\big) &\implies Y_1 \subseteq f(f^{-1}(Y_1)) \\
Y_1 \subseteq f(f^{-1}(Y_1)) \wedge f(f^{-1}(Y_1)) \subseteq Y_1 &\implies Y_1 = f(f^{-1}(Y_1)) \\
\therefore f \text{ is surjective} &\implies Y_1 = f(f^{-1}(Y_1))
\end{aligned}$$

$$\begin{aligned}
\text{Assume } f \text{ is not surjective} &\implies \exists y \in Y : (\nexists x \in X : y = f(x)) \\
&\implies \exists y \in Y : \big(\nexists x \in X : x = f^{-1}(y)\big) \\
&\implies \exists y \in Y : \big(\nexists x \in X : f(x) = f(f^{-1}(y))\big) \\
&\implies \exists y \in Y : f(f^{-1}(\{y\})) = \emptyset \\
&\implies \exists Y_1 \subset Y : Y_1 \neq f(f^{-1}(Y_1)) \\
&\implies \neg\big(Y_1 = f(f^{-1}(Y_1))\ \forall Y_1 \subset Y\big) \\
\therefore Y_1 = f(f^{-1}(Y_1))\ \forall Y_1 \subset Y &\implies f \text{ is surjective (by contraposition)}
\end{aligned}$$

Thus f is surjective $\iff Y_1 = f(f^{-1}(Y_1))\ \forall Y_1 \subset Y$

3.

$$(g \circ f)(x) \equiv g(f(x)) \; \forall x \in X \iff (g \circ f)(X_1) = g(f(X_1)) \; \forall X_1 \subset X$$

$$\begin{aligned}
(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ ((f \circ f^{-1}) \circ g^{-1}) \\
&= g \circ (\mathrm{id}_Y \circ g^{-1}) \\
&= g \circ g^{-1} \\
&= \mathrm{id}_Z \\
(f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ ((g^{-1} \circ g) \circ f) \\
&= f^{-1} \circ (\mathrm{id}_Y \circ f) \\
&= f^{-1} \circ f \\
&= \mathrm{id}_X \\
\implies (g \circ f)^{-1} &= (f^{-1} \circ g^{-1}) \\
\therefore (g \circ f)^{-1}(Y_1) &= (f^{-1} \circ g^{-1})(Y_1) \\
&= f^{-1}(g^{-1}(Y_1))
\end{aligned}$$

4.

$f : X \mapsto Y, \; g : Y \mapsto Z$

$$\begin{aligned}
\text{Assume } f \text{ is not injective} &\implies \exists x, x' \in X : f(x) = f(x') \land x \neq x' \\
&\implies \exists x, x' \in X : g(f(x)) = g(f(x')) \land x \neq x' \\
&\implies \exists x, x' \in X : (g \circ f)(x) = (g \circ f)(x') \land x \neq x' \\
&\implies (g \circ f) \text{ is not injective} \\
\therefore (g \circ f) \text{ is injective} &\implies f \text{ is injective (by contraposition)}
\end{aligned}$$

$$\begin{aligned}
\text{Assume } g \text{ is not surjective} &\implies \exists z \in Z : (\nexists y \in Y : g(y) = z) \\
&\implies \exists z \in Z : (\nexists x \in X : g(f(x)) = z) \\
&\implies \exists z \in Z : (\nexists x \in X : (g \circ f)(x) = z) \\
&\implies (g \circ f) \text{ is not surjective} \\
\therefore (g \circ f) \text{ is surjective} &\implies g \text{ is surjective (by contraposition)}
\end{aligned}$$

Problem 3

1.

$$\begin{aligned}
a < b \land c > 0 &\implies ac < bc \\
c < d \land b > 0 &\implies bc < bd \\
(ac < bc) \land (bc < bd) &\implies ac < bd \; \forall a, b, c, d \in F : 0 < a < b \land 0 < c < d
\end{aligned}$$

2.

$$
\begin{aligned}
a^2+b^2=0 &\iff a^2+b^2-b^2=0-b^2\\
&\iff a^2=-b^2\\
\left(a^2\geq 0\wedge -b^2\leq 0\right)\wedge\left(a^2=-b^2\right) &\implies a^2=0\\
&\implies a=0\\
a^2=-b^2 &\implies b^2=0\\
&\implies b=0\\
a=0\wedge b=0 &\implies b=a\\
\therefore a^2+b^2=0 &\implies a=0\wedge b=a
\end{aligned}
$$

3.

$$
\begin{aligned}
\left(\frac{a-b}{2}\right)^2\geq 0\ \forall a,b\in F &\iff \frac{a^2}{2^2}+\frac{b^2}{2^2}-\frac{ab}{2}\geq 0\ \forall a,b\in F\\
&\iff \frac{a^2}{2^2}+\frac{b^2}{2^2}-\frac{ab}{2}+ab\geq 0+ab\ \forall a,b\in F\\
&\iff \frac{a^2}{2^2}+\frac{b^2}{2^2}+\frac{ab}{2}\geq ab\ \forall a,b\in F\\
&\iff \left(\frac{a+b}{2}\right)^2\geq ab\ \forall a,b\in F
\end{aligned}
$$

Problem 4

$$
\begin{aligned}
F\text{ is of characteristic }p &\iff 1+1+\ldots+1=0\\
&\iff (p-1)+1=0\\
&\implies \exists 0\leq a\in F: a+1=0\\
&\implies F\text{ cannot be an ordered field}
\end{aligned}
$$