MAU23401: Advanced Classical Mechanics I
Homework 9 due 11/12/2020

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SF Theoretical Physics

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Problem 1

(a)

$$
\begin{aligned}
I_{ij} &= \int_G dV\,\rho\left(\vec{r}\right)\left(\delta_{ij}x_n^2 - x_i x_j\right) \\
&= \int_G dV\,\frac{3M}{4\pi R^5}\left(R^2 + \frac{x^2 - y^2}{3} - \frac{2xy}{\sqrt{3}} + xz - \frac{yz}{\sqrt{3}}\right)\left(\delta_{ij}x_n^2 - x_i x_j\right) \\
&= \begin{pmatrix} \frac{8MR^2}{21} & \frac{2MR^2}{35\sqrt{3}} & -\frac{MR^2}{35} \\ \frac{2MR^2}{35\sqrt{3}} & \frac{44MR^2}{105} & \frac{MR^2}{35\sqrt{3}} \\ -\frac{MR^2}{35} & \frac{MR^2}{35\sqrt{3}} & \frac{2MR^2}{5} \end{pmatrix} \\
I_{ij} &= \begin{pmatrix} \frac{20}{21} & \frac{1}{7\sqrt{3}} & -\frac{1}{14} \\ \frac{1}{7\sqrt{3}} & \frac{22}{21} & \frac{1}{14\sqrt{3}} \\ -\frac{1}{14} & \frac{1}{14\sqrt{3}} & 1 \end{pmatrix} I(M,R), \qquad I(M,R) = \frac{2MR^2}{5}
\end{aligned}
$$

```
$Assumptions = R > 0;
```

$$\bar{\rho} = \frac{3\,M}{4\,\pi\,R^3};$$

$$\rho[x\_, y\_, z\_] = \frac{\bar{\rho}}{R^2}\left(R^2 + \frac{x^2 - y^2}{3} - \frac{2\,x\,y}{\sqrt{3}} + x\,z - \frac{y\,z}{\sqrt{3}}\right)$$

$$\frac{3\,M\left(R^2 - \frac{2\,x\,y}{\sqrt{3}} + \frac{1}{3}\left(x^2 - y^2\right) + x\,z - \frac{y\,z}{\sqrt{3}}\right)}{4\,\pi\,R^5}$$

```
ixx = Integrate[ρ[x, y, z] (y^2 + z^2), {x, y, z} ∈ Ball[{0, 0, 0}, R]]
iyy = Integrate[ρ[x, y, z] (x^2 + z^2), {x, y, z} ∈ Ball[{0, 0, 0}, R]]
izz = Integrate[ρ[x, y, z] (x^2 + y^2), {x, y, z} ∈ Ball[{0, 0, 0}, R]]
ixy = Integrate[ρ[x, y, z] (-x y), {x, y, z} ∈ Ball[{0, 0, 0}, R]]
ixz = Integrate[ρ[x, y, z] (-x z), {x, y, z} ∈ Ball[{0, 0, 0}, R]]
iyz = Integrate[ρ[x, y, z] (-y z), {x, y, z} ∈ Ball[{0, 0, 0}, R]]
```

$$\frac{8\,M\,R^2}{21}$$

$$\frac{44\,M\,R^2}{105}$$

$$\frac{2\,M\,R^2}{5}$$

$$\frac{2\,M\,R^2}{35\,\sqrt{3}}$$

$$-\frac{M\,R^2}{35}$$

$$\frac{M\,R^2}{35\,\sqrt{3}}$$

(b)

We can diagonalise the inertia tensor without having to find the diagonalising matrix by simply finding the eigenvalues, and forming a diagonal matrix with these eigenvalues such that $I_{\bar{x}} \le I_{\bar{z}} \le I_{\bar{y}}$.

$$\begin{aligned}
\lambda_{1,2,3} &= \frac{23}{21}, \frac{22}{21}, \frac{6}{7} \\
\mathcal{I} &= \mathrm{diag}\left(\lambda_3, \lambda_1, \lambda_2\right) I(M,R) \\
&= \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & \frac{23}{21} & 0 \\ 0 & 0 & \frac{22}{21} \end{pmatrix} I(M,R) \\
\implies I_{\bar{x},\bar{y},\bar{z}} &= \frac{6}{7} I(M,R),\ \frac{23}{21} I(M,R),\ \frac{22}{21} I(M,R)
\end{aligned}$$

The principal axes will simply be the corresponding eigenvectors.

$$\text{Principal axes } \{\bar{x}, \bar{y}, \bar{z}\} = \left\{ \begin{pmatrix} \frac{3}{2} \\ -\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2\sqrt{3}} \\ 1 \end{pmatrix} \right\} I(M,R)$$

$$\mathbf{i_{tensor'}} = \mathbf{DiagonalMatrix}\left[\mathbf{Eigenvalues}\left[\begin{pmatrix} \frac{20}{21} & \frac{1}{7\sqrt{3}} & -\frac{1}{14} \\ \frac{1}{7\sqrt{3}} & \frac{22}{21} & \frac{1}{14\sqrt{3}} \\ -\frac{1}{14} & \frac{1}{14\sqrt{3}} & 1 \end{pmatrix}\right]\right] \mathbin{//} \mathbf{MatrixForm}$$

ixForm=

$$\begin{pmatrix} \frac{23}{21} & 0 & 0 \\ 0 & \frac{22}{21} & 0 \\ 0 & 0 & \frac{6}{7} \end{pmatrix}$$

$$\mathbf{Eigenvectors}\left[\begin{pmatrix} \frac{20}{21} & \frac{1}{7\sqrt{3}} & -\frac{1}{14} \\ \frac{1}{7\sqrt{3}} & \frac{22}{21} & \frac{1}{14\sqrt{3}} \\ -\frac{1}{14} & \frac{1}{14\sqrt{3}} & 1 \end{pmatrix}\right]$$

$$\left\{\left\{\frac{1}{\sqrt{3}}, 1, 0\right\}, \left\{-\frac{1}{2}, \frac{1}{2\sqrt{3}}, 1\right\}, \left\{\frac{3}{2}, -\frac{\sqrt{3}}{2}, 1\right\}\right\}$$

(c)

$$\text{Unit principal axes } \{\hat{\tilde{x}}, \hat{\tilde{y}}, \hat{\tilde{z}}\} = \left\{ \begin{pmatrix} \frac{3}{4} \\ -\frac{\sqrt{3}}{4} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{4} \\ \frac{1}{4} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \right\}$$

$$\texttt{Normalize}\left[\left\{\frac{3}{2}, -\frac{\sqrt{3}}{2}, 1\right\}\right]$$

$$\texttt{Normalize}\left[\left\{\frac{1}{\sqrt{3}}, 1, 0\right\}\right]$$

$$\texttt{Normalize}\left[\left\{-\frac{1}{2}, \frac{1}{2\sqrt{3}}, 1\right\}\right]$$

$$\left\{\frac{3}{4}, -\frac{\sqrt{3}}{4}, \frac{1}{2}\right\}$$

$$\left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right\}$$

$$\left\{-\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right\}$$

(d)

Let $\hat{v}$ point in the direction of the axis. Small oscillations of this ball about the given axis is effectively a compound pendulum.

$$\omega^2 = \frac{mgl}{ml^2 + I_{\bar{x}} \cos^2\alpha + I_{\bar{y}} \cos^2\beta + I_{\bar{z}} \cos^2\gamma}$$

$$\cos\alpha = \hat{\bar{x}} \cdot \hat{v} = \begin{pmatrix} \frac{3}{4} \\ -\frac{\sqrt{3}}{4} \\ \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \cos\zeta \\ \sin\zeta \\ 0 \end{pmatrix} = \frac{3\cos\zeta - \sqrt{3}\sin\zeta}{4}$$

$$\cos\beta = \hat{\bar{y}} \cdot \hat{v} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\zeta \\ \sin\zeta \\ 0 \end{pmatrix} = \frac{\cos\zeta + \sqrt{3}\sin\zeta}{2}$$

$$\cos\gamma = \hat{\bar{z}} \cdot \hat{v} = \begin{pmatrix} -\frac{\sqrt{3}}{4} \\ \frac{1}{4} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \cdot \begin{pmatrix} \cos\zeta \\ \sin\zeta \\ 0 \end{pmatrix} = \frac{-\sqrt{3}\cos\zeta + \sin\zeta}{4}$$

$$\implies \omega_{\text{inhomo}} = \sqrt{\frac{105g}{R\left(147 - 2\cos(2\zeta) + 2\sqrt{3}\sin(2\zeta)\right)}}$$

$$\cos\alpha = \cos\zeta \qquad I_{\bar{x}} = \frac{2MR^2}{5}$$

$$\cos\beta = \sin\zeta \qquad I_{\bar{y}} = \frac{2MR^2}{5}$$

$$\cos\gamma = 0 \qquad I_{\bar{z}} = \frac{2MR^2}{5}$$

$$\implies \omega_{\text{homo}} = \sqrt{\frac{5g}{7R}}$$

$$\omega_{\text{inhomo}} = \omega_{\text{homo}}$$

$$\implies \zeta = \left\{-\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}\right\} + 2c,\ c \in \mathbb{Z}$$

```
COMx = Integrate[1/M x ρ[x, y, z], {x, y, z} ∈ Ball[{0, 0, 0}, R]]
COMy = Integrate[1/M y ρ[x, y, z], {x, y, z} ∈ Ball[{0, 0, 0}, R]]
COMz = Integrate[1/M z ρ[x, y, z], {x, y, z} ∈ Ball[{0, 0, 0}, R]]
0
0
0
```

$$\cos_{\alpha}[\xi\_] = \left\{\frac{3}{4}, -\frac{\sqrt{3}}{4}, \frac{1}{2}\right\}.\{\mathrm{Cos}[\xi], \mathrm{Sin}[\xi], 0\}$$

$$\cos_{\beta}[\xi\_] = \left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right\}.\{\mathrm{Cos}[\xi], \mathrm{Sin}[\xi], 0\}$$

$$\cos_{\gamma}[\xi\_] = \left\{-\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right\}.\{\mathrm{Cos}[\xi], \mathrm{Sin}[\xi], 0\}$$

$$\frac{3\,\mathrm{Cos}[\xi]}{4} - \frac{1}{4}\sqrt{3}\,\mathrm{Sin}[\xi]$$

$$\frac{\mathrm{Cos}[\xi]}{2} + \frac{1}{2}\sqrt{3}\,\mathrm{Sin}[\xi]$$

$$-\frac{1}{4}\sqrt{3}\,\mathrm{Cos}[\xi] + \frac{\mathrm{Sin}[\xi]}{4}$$

$$i_{M,R} = \frac{2\,M\,R^2}{5};$$

$$\omega_{inhomo}[\xi\_] = \sqrt{\left((M\,g\,R) \Big/ \left(M\,R^2 + \frac{6\,i_{M,R}}{7}\left(\frac{3\,\mathrm{Cos}[\xi]}{4} - \frac{1}{4}\sqrt{3}\,\mathrm{Sin}[\xi]\right)^2 + \frac{23\,i_{M,R}}{21}\left(\frac{\mathrm{Cos}[\xi]}{2} + \frac{1}{2}\sqrt{3}\,\mathrm{Sin}[\xi]\right)^2 + \frac{22\,i_{M,R}}{21}\left(-\frac{1}{4}\sqrt{3}\,\mathrm{Cos}[\xi] + \frac{\mathrm{Sin}[\xi]}{4}\right)^2\right)\right)}\ \text{// FullSimplify}$$

$$\sqrt{105}\sqrt{\frac{g}{R\left(147 - 2\,\mathrm{Cos}[2\,\xi] + 2\sqrt{3}\,\mathrm{Sin}[2\,\xi]\right)}}$$

$$\omega_{homo}[\xi\_] = \sqrt{\frac{M\,g\,R}{M\,R^2 + \frac{2\,M\,R^2}{5}(\mathrm{Cos}[\xi])^2 + \frac{2\,M\,R^2}{5}(\mathrm{Sin}[\xi])^2}}\ \text{// FullSimplify}$$

$$\sqrt{\frac{5}{7}}\sqrt{\frac{g}{R}}$$

$$\mathrm{Solve}\left[\left(M\,R^2 + \frac{6\,i_{M,R}}{7}\left(\frac{3\,\mathrm{Cos}[\xi]}{4} - \frac{1}{4}\sqrt{3}\,\mathrm{Sin}[\xi]\right)^2 + \frac{23\,i_{M,R}}{21}\left(\frac{\mathrm{Cos}[\xi]}{2} + \frac{1}{2}\sqrt{3}\,\mathrm{Sin}[\xi]\right)^2 + \frac{22\,i_{M,R}}{21}\left(-\frac{1}{4}\sqrt{3}\,\mathrm{Cos}[\xi] + \frac{\mathrm{Sin}[\xi]}{4}\right)^2\right) == M\,R^2 + \frac{2\,M\,R^2}{5}, \xi\right]\ \text{// FullSimplify}$$

$$\left\{\left\{\xi \to \pi\left(-\frac{5}{12} + 2\,c_1\right) \text{ if } c_1 \in \mathbb{Z}\right\}, \left\{\xi \to \pi\left(\frac{7}{12} + 2\,c_1\right) \text{ if } c_1 \in \mathbb{Z}\right\}, \left\{\xi \to \pi\left(\frac{1}{12} + 2\,c_1\right) \text{ if } c_1 \in \mathbb{Z}\right\}, \left\{\xi \to \pi\left(-\frac{11}{12} + 2\,c_1\right) \text{ if } c_1 \in \mathbb{Z}\right\}\right\}$$

Problem 2

The tensor of inertia $I_{ik}^{(G)}$ of G defined with respect to its centre of mass O_G satisfies the additivity property

$$I_{ik}^{(G)} = I_{ik}^{(A)}\left(\vec{a}_A\right) + I_{ik}^{(B)}\left(\vec{a}_B\right),$$

where $I_{ik}^{(A)}\left(\vec{a}_A\right)$ is the inertia tensor $I_{ik}^{(A)}$ of A defined with respect to O_G, and $\vec{a}_A$ is the vector from O_G to O_A, and similarly for B.

The tensors $I_{ik}^{(A)}\left(\vec{a}_A\right)$ and $I_{ik}^{(B)}\left(\vec{a}_B\right)$ are given by

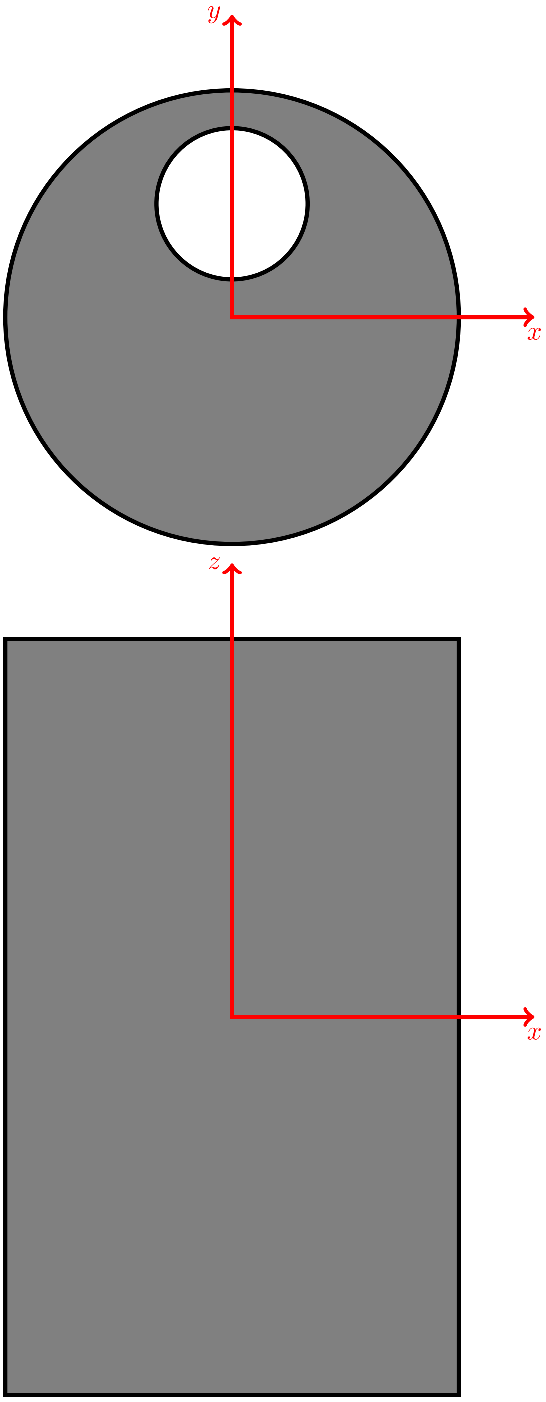
$$I_{ik}\left(\vec{a}\right) = I_{ik} + m\left(\vec{a}^2\delta_{ik} - a_i a_k\right),$$

and therefore

$$\begin{aligned} & I_{ik}^{(G)} = I_{ik}^{(A)} + I_{ik}^{(B)} + m_A\left(a_A^2\delta_{ik} - a_{Ai}a_{Ak}\right) + m_B\left(a_B^2\delta_{ik} - a_{Bi}a_{Bk}\right) \\ \Longrightarrow\ & I_{ik}^{(A)} = I_{ik}^{(G)} - I_{ik}^{(B)} - m_A\left(a_A^2\delta_{ik} - a_{Ai}a_{Ak}\right) - m_B\left(a_B^2\delta_{ik} - a_{Bi}a_{Bk}\right) \end{aligned}$$

Problem 3

(a)



$$
\begin{aligned}
I_{ik}^{(A)} &= I_{ik}^{(G)} - I_{ik}^{(B)} - m_A\left(a_A^2\delta_{ik} - a_{Ai}a_{Ak}\right) - m_B\left(a_B^2\delta_{ik} - a_{Bi}a_{Bk}\right) \\
m_B &= \rho\pi r^2 H \\
m_A &= m_G - m_B \\
&= \rho\pi H\left(R^2 - r^2\right) \\
a_B &= a \\
a_A &= \frac{m_G(0) - m_B(a)}{m_G - m_B} \\
&= -\frac{ar^2}{R^2 - r^2} \\
I_{ik}^{(G)} &= \int_{-\frac{H}{2}}^{\frac{H}{2}}\int_{\mathrm{area}_G} \mathrm{darea}_G\, dz\, \rho\left(\delta_{ik}x_n^2 - x_i x_k\right) \\
&= \begin{pmatrix} \frac{1}{12}H^3\pi R^2\rho + \frac{1}{4}H\pi R^4\rho & 0 & 0 \\ 0 & \frac{1}{12}H^3\pi R^2\rho + \frac{1}{4}H\pi R^4\rho & 0 \\ 0 & 0 & \frac{1}{2}H\pi R^4\rho \end{pmatrix} \\
I_{ik}^{(B)} &= \int_{-\frac{H}{2}}^{\frac{H}{2}}\int_{\mathrm{area}_B} \mathrm{darea}_B\, dz\, \rho\left(\delta_{ik}x_n^2 - x_i x_k\right) \\
&= \begin{pmatrix} \frac{1}{12}H^3\pi r^2\rho + \frac{1}{4}H\pi r^4\rho & 0 & 0 \\ 0 & \frac{1}{12}H^3\pi r^2\rho + \frac{1}{4}H\pi r^4\rho & 0 \\ 0 & 0 & \frac{1}{2}H\pi r^4\rho \end{pmatrix} \\
m_A\left(a_A^2\delta_{ik} - a_{Ai}a_{Ak}\right) &= \begin{pmatrix} \frac{a^2H\pi r^4\rho}{R^2-r^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{a^2H\pi r^4\rho}{R^2-r^2} \end{pmatrix} \\
m_B\left(a_B^2\delta_{ik} - a_{Bi}a_{Bk}\right) &= \begin{pmatrix} a^2H\pi r^4\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a^2H\pi r^4\rho \end{pmatrix} \\
\implies I_{\bar{x}} &= \frac{H\pi\rho\left(H^2\left(R^2 - r^2\right)^2 + 3\left(R^6 + r^6 - R^2r^4 - r^2\left(R^4 + 4a^2R^2\right)\right)\right)}{12\left(R^2 - r^2\right)} \\
I_{\bar{y}} &= \frac{1}{12}H\pi\rho\left(R - r\right)\left(R + r\right)\left(H^2 + 3\left(R^2 + r^2\right)\right) \\
I_{\bar{z}} &= \frac{H\pi\rho\left(R^6 + r^6 - R^2r^4 - r^2\left(R^4 + 2a^2R^2\right)\right)}{2\left(R^2 - r^2\right)}
\end{aligned}
$$

```
$Assumptions = R > 0 && H > 0 && r > 0 && ρ > 0 && a > 0 && a ≤ R - r && r < R;
iGxx = Integrate[Integrate[ρ (y^2 + z^2), {x, y} ∈ Disk[{0, 2}, R]], {z, -H/2, H/2}]
iGyy = Integrate[Integrate[ρ (x^2 + z^2), {x, y} ∈ Disk[{0, 0}, R]], {z, -H/2, H/2}]
iGzz = Integrate[Integrate[ρ (x^2 + y^2), {x, y} ∈ Disk[{0, 0}, R]], {z, -H/2, H/2}]
iGxy = Integrate[Integrate[ρ (-x y), {x, y} ∈ Disk[{0, 0}, R]], {z, -H/2, H/2}]
iGxz = Integrate[Integrate[ρ (-x z), {x, y} ∈ Disk[{0, 0}, R]], {z, -H/2, H/2}]
iGyz = Integrate[Integrate[ρ (-y z), {x, y} ∈ Disk[{0, 0}, R]], {z, -H/2, H/2}]
```

$$4 H \pi R^2 \rho + \frac{1}{12} H^3 \pi R^2 \rho + \frac{1}{4} H \pi R^4 \rho$$

$$\frac{1}{12} H^3 \pi R^2 \rho + \frac{1}{4} H \pi R^4 \rho$$

$$\frac{1}{2} H \pi R^4 \rho$$

0

0

0

```
i_Bxx = Integrate[Integrate[ρ (y^2 + z^2), {x, y} ∈ Disk[{0, 0}, r]], {z, -H/2, H/2}]
i_Byy = Integrate[Integrate[ρ (x^2 + z^2), {x, y} ∈ Disk[{0, 0}, r]], {z, -H/2, H/2}]
i_Bzz = Integrate[Integrate[ρ (x^2 + y^2), {x, y} ∈ Disk[{0, 0}, r]], {z, -H/2, H/2}]
i_Bxy = Integrate[Integrate[ρ (-x y), {x, y} ∈ Disk[{0, 0}, r]], {z, -H/2, H/2}]
i_Bxz = Integrate[Integrate[ρ (-x z), {x, y} ∈ Disk[{0, 0}, r]], {z, -H/2, H/2}]
i_Byz = Integrate[Integrate[ρ (-y z), {x, y} ∈ Disk[{0, 0}, r]], {z, -H/2, H/2}]
```

$$\frac{1}{12} H^3 \pi r^2 \rho + \frac{1}{4} H \pi r^4 \rho$$

$$\frac{1}{12} H^3 \pi r^2 \rho + \frac{1}{4} H \pi r^4 \rho$$

$$\frac{1}{2} H \pi r^4 \rho$$

0

0

0

$$\alpha_{xx} = \rho \pi H \left(R^2 - r^2\right) \left(\left(\frac{a r^2}{R^2 - r^2}\right)^2 - 0\right)$$

$$\alpha_{yy} = \rho \pi H \left(R^2 - r^2\right) \left(\left(\frac{a r^2}{R^2 - r^2}\right)^2 - \left(\frac{a r^2}{R^2 - r^2}\right)^2\right)$$

$$\alpha_{zz} = \rho \pi H \left(R^2 - r^2\right) \left(\left(\frac{a r^2}{R^2 - r^2}\right)^2 - 0\right)$$

$$\alpha_{xy} = \rho \pi H \left(R^2 - r^2\right) (0 - 0)$$

$$\alpha_{xz} = \rho \pi H \left(R^2 - r^2\right) (0 - 0)$$

$$\alpha_{yz} = \rho \pi H \left(R^2 - r^2\right) (0 - 0)$$

$$\frac{a^2 H \pi r^4 \rho}{-r^2 + R^2}$$

$$0$$

$$\frac{a^2 H \pi r^4 \rho}{-r^2 + R^2}$$

$$0$$

$$0$$

$$0$$

$$\beta_{xx} = \rho \pi r^2 H \left(a^2 - 0\right)$$

$$\beta_{yy} = \rho \pi r^2 H \left(a^2 - a^2\right)$$

$$\beta_{zz} = \rho \pi r^2 H \left(a^2 - 0\right)$$

$$\beta_{xy} = \rho \pi r^2 H (0 - 0)$$

$$\beta_{xz} = \rho \pi r^2 H (0 - 0)$$

$$\beta_{yz} = \rho \pi r^2 H (0 - 0)$$

$$a^2 H \pi r^2 \rho$$

$$0$$

$$a^2 H \pi r^2 \rho$$

$$0$$

$$0$$

$$0$$

In[ ]:= $\left\{\left\{\frac{1}{12} H^3 \pi R^2 \rho + \frac{1}{4} H \pi R^4 \rho - \left(\frac{1}{12} H^3 \pi r^2 \rho + \frac{1}{4} H \pi r^4 \rho\right) - \left(\frac{a^2 H \pi r^4 \rho}{-r^2 + R^2}\right) - \left(a^2 H \pi r^2 \rho\right), 0, 0\right\},\right.$
$\left\{0, \frac{1}{12} H^3 \pi R^2 \rho + \frac{1}{4} H \pi R^4 \rho - \left(\frac{1}{12} H^3 \pi r^2 \rho + \frac{1}{4} H \pi r^4 \rho\right) - (0) - (0), 0\right\},$
$\left.\left\{0, 0, \frac{1}{2} H \pi R^4 \rho - \left(\frac{1}{2} H \pi r^4 \rho\right) - \left(\frac{a^2 H \pi r^4 \rho}{-r^2 + R^2}\right) - \left(a^2 H \pi r^2 \rho\right)\right\}\right\}$ // FullSimplify // MatrixForm

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{H \pi \left(H^2 \left(r^2 - R^2\right)^2 + 3 \left(r^6 - r^4 R^2 + R^6 - r^2 \left(4 a^2 R^2 + R^4\right)\right)\right) \rho}{12 \left(r^2 - R^2\right)} & 0 & 0 \\ 0 & -\frac{1}{12} H \pi (r - R) (r + R) \left(H^2 + 3 \left(r^2 + R^2\right)\right) \rho & 0 \\ 0 & 0 & -\frac{H \pi \left(r^6 - r^4 R^2 + R^6 - r^2 \left(2 a^2 R^2 + R^4\right)\right) \rho}{2 \left(r^2 - R^2\right)} \end{pmatrix}$$

(b)

The xz-plane is considered as the horizontal plane. Small oscillations of this cylinder about the given axis is effectively a compound pendulum. $\alpha = \beta = \frac{\pi}{2}$, $\gamma = 0$.

$$\omega^2 = \frac{mgl}{ml^2 + I_{\bar{x}} \cos^2 \alpha + I_{\bar{y}} \cos^2 \beta + I_{\bar{z}} \cos^2 \gamma}$$

$$\omega = \sqrt{\frac{\rho \pi H \left(R^2 - r^2\right) gl}{\rho \pi H \left(R^2 - r^2\right) l^2 + I_{\bar{z}}}}$$

$$l_G = \frac{ar^2}{R^2 - r^2} \qquad l_B = \frac{ar^2}{R^2 - r^2} + a$$

$$\omega_G = \sqrt{\frac{2agr^2}{R^4 - r^4 - 2a^2r^2}} \qquad \omega_B = \sqrt{\frac{2agR^2}{R^4 - r^4 + 2a^2R^2}}$$

$$\omega_G = \sqrt{\frac{\rho \pi H \left(R^2 - r^2\right) g \left(\frac{a r^2}{R^2 - r^2}\right)}{\rho \pi H \left(R^2 - r^2\right) \left(\frac{a r^2}{R^2 - r^2}\right)^2 - \frac{H \pi \left(r^6 - r^4 R^2 + R^6 - r^2 \left(2 a^2 R^2 + R^4\right)\right) \rho}{2 \left(r^2 - R^2\right)}}} \;\; \text{// FullSimplify}$$

$$\sqrt{2}\, r \sqrt{\frac{a g}{-2 a^2 r^2 - r^4 + R^4}}$$

$$\omega_B = \sqrt{\frac{\rho \pi H \left(R^2 - r^2\right) g \left(\frac{a r^2}{R^2 - r^2} + a\right)}{\rho \pi H \left(R^2 - r^2\right) \left(\frac{a r^2}{R^2 - r^2} + a\right)^2 - \frac{H \pi \left(r^6 - r^4 R^2 + R^6 - r^2 \left(2 a^2 R^2 + R^4\right)\right) \rho}{2 \left(r^2 - R^2\right)}}} \;\; \text{// FullSimplify}$$

$$\sqrt{2} \sqrt{\frac{a g R^2}{-r^4 + 2 a^2 R^2 + R^4}}$$