MAU23401: Advanced Classical Mechanics I
Homework 8 due 04/12/2020

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Problem 1

1.

$$\begin{aligned} L &= \frac{m}{2}\vec{v}^2 - U(r) \\ &= \frac{m}{2}\left(x'(t)^2 + y'(t)^2 + z'(t)^2\right) - mg\,z(t) \\ z(t) &= \frac{k}{2}\left(x^2+y^2\right) \\ \implies L &= \frac{m}{2}\left(x'(t)^2 + y'(t)^2 + k^2\left(x(t)x'(t) + y(t)y'(t)\right)^2 - gk\left(x(t)^2+y(t)^2\right)\right) \end{aligned}$$

Since swapping x and y results in no change to the Lagrangian, we can solve the equations of motion in terms of x only, and simply swap x for y to get our answer in terms of y.

$$\begin{aligned} \text{eom: } 0 &= \frac{d}{dt}\frac{\partial L}{\partial x'(t)} - \frac{\partial L}{\partial x(t)} + \eta\,x'(t) \\ \implies x''(t) &= -\frac{\eta\,x'(t) + km\,x(t)\left(g + k\left(x'(t)^2 + y'(t)^2 + y(t)y''(t)\right)\right)}{m + k^2 m\,x(t)^2} \\ \implies y''(t) &= -\frac{\eta\,y'(t) + km\,y(t)\left(g + k\left(x'(t)^2 + y'(t)^2 + x(t)x''(t)\right)\right)}{m + k^2 m\,y(t)^2} \end{aligned}$$

```
L = m/2 (x'[t]^2 + y'[t]^2 + z'[t]^2) - m g z[t];
z[t_] = k/2 (x[t]^2 + y[t]^2);
L // FullSimplify
```

$$\frac{1}{2} m \left(-g k \left(x[t]^2 + y[t]^2\right) + x'[t]^2 + y'[t]^2 + k^2 \left(x[t]\, x'[t] + y[t]\, y'[t]\right)^2\right)$$

```
EOMx = D[D[L, x'[t]], t] - D[L, x[t]] + η x'[t];
Solve[EOMx == 0, x''[t]] // FullSimplify
```

$$\left\{\left\{x''[t] \to -\frac{\eta\, x'[t] + k\, m\, x[t] \left(g + k \left(x'[t]^2 + y'[t]^2 + y[t]\, y''[t]\right)\right)}{m + k^2\, m\, x[t]^2}\right\}\right\}$$

2.

Like before, we will only do this question with regards to x, and then write a similar answer for y based on this.

If we are only considering small oscillations then all quadratic terms will be approximately 0 compared to the linear terms.

$$\begin{aligned}
x''(t) &\approx -\frac{\eta\, x'(t) + gkm\, x(t)}{m} \\
\text{ansatz: } x(t) &= \Re\left(Ae^{rt}\right) \\
\implies r^2 &= -\frac{\eta r + gkm}{m} \\
\implies r_\pm &= \frac{1}{2}\left(-\frac{\eta}{m} \pm \frac{\sqrt{\eta^2 - 4gkm^2}}{m}\right) \\
x(t) &= \Re\left(A_+ e^{r_+ t} + A_- e^{r_- t}\right) \\
&= e^{-\frac{\eta t}{2m}} \Re\left(A_+ e^{\frac{t}{2m}\sqrt{\eta^2-4gkm^2}} + A_- e^{-\frac{t}{2m}\sqrt{\eta^2-4gkm^2}}\right)
\end{aligned}$$

$$\begin{aligned}
&\text{Case 1: } \eta^2 < 4gkm^2 \\
\implies x(t) &= e^{-\frac{\eta t}{2m}} \Re\left(A_+ e^{\frac{it}{2m}\sqrt{4gkm^2-\eta^2}} + A_- e^{-\frac{it}{2m}\sqrt{4gkm^2-\eta^2}}\right) \\
&= e^{-\frac{\eta t}{2m}} \Re\left(A_+ \cos\left(\frac{t}{2m}\sqrt{4gkm^2 - \eta^2}\right) + A_- \cos\left(-\frac{t}{2m}\sqrt{4gkm^2 - \eta^2}\right)\right) \\
&= \alpha e^{-\frac{\eta t}{2m}} \cos\left(\frac{t}{2m}\sqrt{4gkm^2 - \eta^2}\right), \quad \alpha \equiv \Re\left(A_+ + A_-\right)
\end{aligned}$$

$$\begin{aligned}
&\text{Case 2: } \eta^2 > 4gkm^2 \\
\implies x(t) &= e^{-\frac{\eta t}{2m}} \Re\left(A_+ e^{\frac{t}{2m}\sqrt{\eta^2-4gkm^2}} + A_- e^{-\frac{t}{2m}\sqrt{\eta^2-4gkm^2}}\right) \\
&= e^{-\frac{\eta t}{2m}} \left(\alpha_+ e^{\frac{t}{2m}\sqrt{\eta^2-4gkm^2}} + \alpha_- e^{-\frac{t}{2m}\sqrt{\eta^2-4gkm^2}}\right), \quad \alpha_\pm \equiv \Re\left(A_\pm\right)
\end{aligned}$$

$$\begin{aligned}
&\text{Case 3: } \eta^2 = 4gkm^2 \\
\implies x(t) &= e^{-\frac{\eta t}{2m}} \Re\left(A_+ + A_-\right) \\
&= \alpha e^{-\frac{\eta t}{2m}}, \quad \alpha \equiv \Re\left(A_+ + A_-\right)
\end{aligned}$$

We can similarly define y, by letting $x \to y$, $A \to B$, $\alpha \to \beta$.

$$\begin{aligned}
\text{Underdamped: } x(t) &= \alpha e^{-\frac{\eta t}{2m}} \cos\left(\frac{t}{2m}\sqrt{4gkm^2 - \eta^2}\right) \\
\text{and } y(t) &= \beta e^{-\frac{\eta t}{2m}} \cos\left(\frac{t}{2m}\sqrt{4gkm^2 - \eta^2}\right) \\
\text{Overdamped: } x(t) &= e^{-\frac{\eta t}{2m}} \left(\alpha_+ e^{\frac{t}{2m}\sqrt{\eta^2-4gkm^2}} + \alpha_- e^{-\frac{t}{2m}\sqrt{\eta^2-4gkm^2}}\right) \\
\text{and } y(t) &= e^{-\frac{\eta t}{2m}} \left(\beta_+ e^{\frac{t}{2m}\sqrt{\eta^2-4gkm^2}} + \beta_- e^{-\frac{t}{2m}\sqrt{\eta^2-4gkm^2}}\right) \\
\text{Critically damped: } x(t) &= \alpha e^{-\frac{\eta t}{2m}} \\
\text{and } y(t) &= \beta e^{-\frac{\eta t}{2m}}
\end{aligned}$$

$$\text{Solve}\left[r^2 + \frac{\eta\, r + g\, k\, m}{m} == 0,\ r\right]$$

$$\left\{\left\{r \to \frac{1}{2}\left(-\frac{\eta}{m} - \frac{\sqrt{-4\, g\, k\, m^2 + \eta^2}}{m}\right)\right\},\ \left\{r \to \frac{1}{2}\left(-\frac{\eta}{m} + \frac{\sqrt{-4\, g\, k\, m^2 + \eta^2}}{m}\right)\right\}\right\}$$

$$r_+ = \frac{1}{2}\left(-\frac{\eta}{m} + \frac{\sqrt{-4\, g\, k\, m^2 + \eta^2}}{m}\right);$$

$$r_- = \frac{1}{2}\left(-\frac{\eta}{m} - \frac{\sqrt{-4\, g\, k\, m^2 + \eta^2}}{m}\right);$$

$$x[t\_] = A_+\, e^{r_+\, t} + A_-\, e^{r_-\, t}$$

$$e^{\frac{1}{2}\, t\left(-\frac{\eta}{m} - \frac{\sqrt{-4\, g\, k\, m^2+\eta^2}}{m}\right)} A_- + e^{\frac{1}{2}\, t\left(-\frac{\eta}{m} + \frac{\sqrt{-4\, g\, k\, m^2+\eta^2}}{m}\right)} A_+$$

Problem 2

1.

$$\begin{aligned}
\text{eom}_{\text{homo}}\text{: } 0 &= x''_{\text{homo}}(t) + 2\lambda x'_{\text{homo}}(t) + {\omega_0}^2 x_{\text{homo}}(t)\\
\implies x_{\text{homo}}(t) &= e^{\left(-\lambda-\sqrt{\lambda^2-{\omega_0}^2}\right)t}c_1 + e^{\left(-\lambda+\sqrt{\lambda^2-{\omega_0}^2}\right)t}c_2\\
\text{eom: } 0 &= x''(t) + 2\lambda x'(t) + {\omega_0}^2 x(t) - f_0 e^{-\alpha|t|}, \quad f_0 = \frac{F_0}{m}\\
t<0 : \text{ansatz: } x(t) &= \Re\left(Ae^{\alpha t}\right)\\
\implies A &= \frac{f_0}{\alpha^2+2\alpha\lambda+{\omega_0}^2}\\
x_-(t) &= x_{\text{homo}}(t) + x(t)\\
&= e^{\left(-\lambda-\sqrt{\lambda^2-{\omega_0}^2}\right)t}c_1 + e^{\left(-\lambda+\sqrt{\lambda^2-{\omega_0}^2}\right)t}c_2 + \Re\left(\frac{f_0}{\alpha^2+2\alpha\lambda+{\omega_0}^2}e^{\alpha t}\right)
\end{aligned}$$

If $t = -\infty$ then $x = \infty$ and thus $E \neq 0$, thus $c_1, c_2 = 0$.

$$\begin{aligned}
\implies x_-(t) &= \Re\left(\frac{f_0}{\alpha^2+2\alpha\lambda+{\omega_0}^2}e^{\alpha t}\right)\\
&= \frac{f_0}{\alpha^2+2\alpha\lambda+{\omega_0}^2}e^{\alpha t}\\
x_-(0) &= \frac{f_0}{\alpha^2+2\alpha\lambda+{\omega_0}^2}\\
{x_-}'(t) &= \frac{\alpha f_0}{\alpha^2+2\alpha\lambda+{\omega_0}^2}\\
t>0 : x(t) &= e^{\left(-\lambda-\sqrt{\lambda^2-{\omega_0}^2}\right)t}d_1 + e^{\left(-\lambda+\sqrt{\lambda^2-{\omega_0}^2}\right)t}d_2 + Be^{-\alpha t}\\
\text{eom: } 0 &= x''(t) + 2\lambda x'(t) + {\omega_0}^2 x(t) - f_0 e^{-\alpha|t|}\\
\implies B &= \frac{f_0}{\alpha^2-2\alpha\lambda+{\omega_0}^2}\\
x_+(t) &= x(t)\\
&= e^{\left(-\lambda-\sqrt{\lambda^2-{\omega_0}^2}\right)t}d_1 + e^{\left(-\lambda+\sqrt{\lambda^2-{\omega_0}^2}\right)t}d_2 + \frac{f_0}{\alpha^2-2\alpha\lambda+{\omega_0}^2}e^{-\alpha t}\\
x_+(0) &= d_1 + d_2 + \frac{f_0}{\alpha^2-2\alpha\lambda+{\omega_0}^2}\\
{x_+}'(t) &= d_1\left(-\lambda-\sqrt{\lambda^2-{\omega_0}^2}\right) + d_2\left(-\lambda+\sqrt{\lambda^2-{\omega_0}^2}\right) - \frac{\alpha f_0}{\alpha^2-2\alpha\lambda+{\omega_0}^2}\\
x_-(0) &= x_+(0)\\
{x_-}'(0) &= {x_+}'(0)\\
\implies d_1 &= -\frac{\alpha f_0\left(\alpha^2+{\omega_0}^2+2\lambda\left(-\lambda+\sqrt{\lambda^2-{\omega_0}^2}\right)\right)}{\sqrt{\lambda^2-{\omega_0}^2}\left(\alpha(\alpha-2\lambda)+{\omega_0}^2\right)\left(\alpha(\alpha+2\lambda)+{\omega_0}^2\right)}\\
\text{and } d_2 &= \frac{\alpha f_0\left(\alpha^2+{\omega_0}^2-2\lambda\left(\lambda+\sqrt{\lambda^2-{\omega_0}^2}\right)\right)}{\sqrt{\lambda^2-{\omega_0}^2}\left(\alpha(\alpha-2\lambda)+{\omega_0}^2\right)\left(\alpha(\alpha+2\lambda)+{\omega_0}^2\right)}
\end{aligned}$$

$$\begin{aligned}
\implies x_+(t) = \frac{f_0 e^{-\alpha t}}{\alpha^2-2\alpha\lambda+{\omega_0}^2} &- \frac{\alpha f_0 e^{\left(-\lambda-\sqrt{\lambda^2-{\omega_0}^2}\right)t}\left(\alpha^2+{\omega_0}^2+2\lambda\left(-\lambda+\sqrt{\lambda^2-{\omega_0}^2}\right)\right)}{\sqrt{\lambda^2-{\omega_0}^2}\left(\alpha(\alpha-2\lambda)+{\omega_0}^2\right)\left(\alpha(\alpha+2\lambda)+{\omega_0}^2\right)}\\
&+ \frac{\alpha f_0\left(\alpha^2+{\omega_0}^2-2\lambda\left(\lambda+\sqrt{\lambda^2-{\omega_0}^2}\right)\right)}{\sqrt{\lambda^2-{\omega_0}^2}\left(\alpha(\alpha-2\lambda)+{\omega_0}^2\right)\left(\alpha(\alpha+2\lambda)+{\omega_0}^2\right)}
\end{aligned}$$

$$x(t) = \begin{cases} x_-(t), & t \leq 0 \\ x_+(t), & t > 0 \end{cases}$$

$$= \begin{cases} \frac{f_0}{\alpha^2+2\alpha\lambda+\omega_0{}^2} e^{\alpha t}, & t \leq 0 \\ \frac{f_0 e^{-\alpha t}}{\alpha^2-2\alpha\lambda+\omega_0{}^2} - \frac{\alpha f_0 e^{\left(-\lambda-\sqrt{\lambda^2-\omega_0{}^2}\right)t}\left(\alpha^2+\omega_0{}^2+2\lambda\left(-\lambda+\sqrt{\lambda^2-\omega_0{}^2}\right)\right)}{\sqrt{\lambda^2-\omega_0{}^2}(\alpha(\alpha-2\lambda)+\omega_0{}^2)(\alpha(\alpha+2\lambda)+\omega_0{}^2)} + \frac{\alpha f_0\left(\alpha^2+\omega_0{}^2-2\lambda\left(\lambda+\sqrt{\lambda^2-\omega_0{}^2}\right)\right)}{\sqrt{\lambda^2-\omega_0{}^2}(\alpha(\alpha-2\lambda)+\omega_0{}^2)(\alpha(\alpha+2\lambda)+\omega_0{}^2)}, & t > 0 \end{cases}$$

```
Clear[x];
$Assumptions = m > 0 && f0 > 0 && α > 0;
EOM1 = x''[t] + 2 λ x'[t] + ω0^2 x[t];
DSolve[EOM1 == 0, x[t], t]
```

$$\left\{\left\{x[t] \to e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)} C_1 + e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)} C_2\right\}\right\}$$

```
Clear[A];
f[t_] = Piecewise[{{f0 e^(α t), t < 0}, {f0 e^(-α t), t ≥ 0}}]
EOM2 = x''[t] + 2 λ x'[t] + ω0^2 x[t] - f[t];
x[t_] = A e^(α t);
Assuming[t < 0, Solve[EOM2 == 0, A]]
```

$$\begin{cases} e^{t\alpha} f0 & t < 0 \\ e^{-t\alpha} f0 & t \geq 0 \\ 0 & \text{True} \end{cases}$$

$$\left\{\left\{A \to \frac{e^{-t\alpha}\left(\begin{cases} e^{t\alpha} f0 & t < 0 \\ e^{-t\alpha} f0 & t \geq 0 \\ 0 & \text{True} \end{cases}\right)}{\alpha^2 + 2\alpha\lambda + \omega_0^2}\right\}\right\}$$

$$x[t\_] = A\, e^{\alpha t};$$

$$A = \frac{f0}{\alpha^2 + 2\,\alpha\,\lambda + \omega_0^2};$$

$$x_{homo}[t\_] = e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)}\, c_1 + e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}\, c_2;$$

$$x_-[t\_] = x_{homo}[t] + x[t]$$

••• Set: Tag Plus in $\left(e^{t\left(-\lambda-\sqrt{\text{Plus[≪2≫]}}\right)}\, c_1 + e^{t\left(-\lambda+\sqrt{\text{Power[≪2≫]+Times[≪2≫]}}\right)}\, c_2\right)$[t_] is Protected.

$$\frac{e^{t\,\alpha}\, f0}{\alpha^2 + 2\,\alpha\,\lambda + \omega_0^2} + \left(e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)}\, c_1 + e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}\, c_2\right)[t]$$

$$x_-[t\_] = x[t]$$

$$\frac{e^{t\,\alpha}\, f0}{\alpha^2 + 2\,\alpha\,\lambda + \omega_0^2}$$

$$X_-[t\_] = \text{Re}[x_-[t]]$$

$$\text{Re}\left[\frac{e^{t\,\alpha}\, f0}{\alpha^2 + 2\,\alpha\,\lambda + \omega_0^2}\right]$$

```
x_0 = X_[0]
```

$$\mathrm{Re}\left[\frac{f0}{\alpha^2 + 2\,\alpha\,\lambda + \omega_0^2}\right]$$

```
v_0 = Re[x_'[0]]
```

$$\mathrm{Re}\left[\frac{f0\,\alpha}{\alpha^2 + 2\,\alpha\,\lambda + \omega_0^2}\right]$$

```
(*t>0*)
Clear[x, B, c1, c2];
```

$$x[t\_] = e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)}\,c_1 + e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}\,c_2 + B\,e^{-\alpha\,t};$$

$$\mathrm{EOM3} = x''[t] + 2\,\lambda\,x'[t] + \omega_0{}^2\,x[t] - f[t];$$

```
Assuming[{t ≥ 0}, EOM3 // FullSimplify]
```

$$e^{-t\,\alpha}\left(-f0 + B\,\alpha\,(\alpha - 2\,\lambda) + B\,\omega_0^2\right)$$

```
Assuming[{t ≥ 0}, Solve[EOM3 == 0, B]]
```

$$\left\{\left\{B \to \frac{e^{t\,\alpha}\left(\begin{cases} e^{t\,\alpha}\,f0 & t < 0 \\ e^{-t\,\alpha}\,f0 & t \geq 0 \\ 0 & \text{True}\end{cases}\right)}{\alpha^2 - 2\,\alpha\,\lambda + \omega_0^2}\right\}\right\}$$

$$B = \frac{f0}{\alpha^2 - 2\alpha\lambda + \omega_0^2};$$

$$x_+[t\_] = x[t]$$

$$e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)} c_1 + e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)} c_2 + \frac{e^{-t\alpha} f0}{\alpha^2 - 2\alpha\lambda + \omega_0^2}$$

$$x_{+0} = x_+[0]$$

$$c_1 + c_2 + \frac{f0}{\alpha^2 - 2\alpha\lambda + \omega_0^2}$$

$$v_{+0} = x_+'[0]$$

$$-\frac{f0\,\alpha}{\alpha^2 - 2\alpha\lambda + \omega_0^2} + c_1\left(-\lambda - \sqrt{\lambda^2-\omega_0^2}\right) + c_2\left(-\lambda + \sqrt{\lambda^2-\omega_0^2}\right)$$

$$\text{Solve}\left[\left\{c_1 + c_2 + \frac{f0}{\alpha^2 - 2\alpha\lambda + \omega_0^2} == \frac{f0}{\alpha^2 + 2\alpha\lambda + \omega_0^2},\right.\right.$$
$$\left.\left. -\frac{f0\,\alpha}{\alpha^2 - 2\alpha\lambda + \omega_0^2} + c_1\left(-\lambda - \sqrt{\lambda^2-\omega_0^2}\right) + c_2\left(-\lambda + \sqrt{\lambda^2-\omega_0^2}\right) == \frac{f0\,\alpha}{\alpha^2 + 2\alpha\lambda + \omega_0^2}\right\}, \{c_1, c_2\}\right] \text{ // FullSimplify}$$

$$\left\{\left\{c_1 \to -\frac{f0\,\alpha\left(\alpha^2 + \omega_0^2 + 2\lambda\left(-\lambda + \sqrt{\lambda^2-\omega_0^2}\right)\right)}{\sqrt{\lambda^2-\omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)},\ c_2 \to \frac{f0\,\alpha\left(\alpha^2 + \omega_0^2 - 2\lambda\left(\lambda + \sqrt{\lambda^2-\omega_0^2}\right)\right)}{\sqrt{\lambda^2-\omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)}\right\}\right\}$$

$$c_1 = -\frac{f0\,\alpha\left(\alpha^2 + \omega_0^2 + 2\lambda\left(-\lambda + \sqrt{\lambda^2-\omega_0^2}\right)\right)}{\sqrt{\lambda^2-\omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)};$$

$$c_2 = \frac{f0\,\alpha\left(\alpha^2 + \omega_0^2 - 2\lambda\left(\lambda + \sqrt{\lambda^2-\omega_0^2}\right)\right)}{\sqrt{\lambda^2-\omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)};$$

$$x_+[t]$$
$$X_+ = \text{Re}[x_+[t]]$$

$$\frac{e^{-t\alpha} f0}{\alpha^2 - 2\alpha\lambda + \omega_0^2} - \frac{e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)} f0\,\alpha\left(\alpha^2 + \omega_0^2 + 2\lambda\left(-\lambda + \sqrt{\lambda^2-\omega_0^2}\right)\right)}{\sqrt{\lambda^2-\omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)} + \frac{e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)} f0\,\alpha\left(\alpha^2 + \omega_0^2 - 2\lambda\left(\lambda + \sqrt{\lambda^2-\omega_0^2}\right)\right)}{\sqrt{\lambda^2-\omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)}$$

$$\text{Re}\left[\frac{e^{-t\alpha} f0}{\alpha^2 - 2\alpha\lambda + \omega_0^2} - \frac{e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)} f0\,\alpha\left(\alpha^2 + \omega_0^2 + 2\lambda\left(-\lambda + \sqrt{\lambda^2-\omega_0^2}\right)\right)}{\sqrt{\lambda^2-\omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)} + \frac{e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)} f0\,\alpha\left(\alpha^2 + \omega_0^2 - 2\lambda\left(\lambda + \sqrt{\lambda^2-\omega_0^2}\right)\right)}{\sqrt{\lambda^2-\omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)}\right]$$

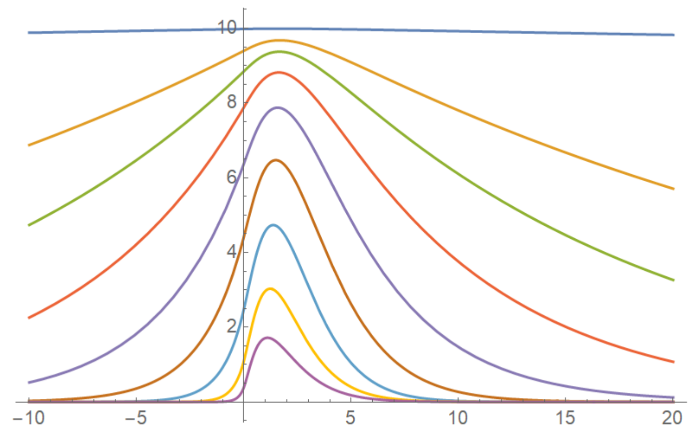
$$X[t\_] = \text{Piecewise}[\{\{x_-[t], t < 0\}, \{x_+[t], t \geq 0\}\}]$$

$$\begin{cases} \frac{e^{t\alpha} f0}{1+\alpha^2+2\alpha\lambda} & t < 0 \\ \frac{e^{-t\alpha} f0}{1+\alpha^2-2\alpha\lambda} - \frac{e^{t\left(-\lambda-\sqrt{-1+\lambda^2}\right)} f0\,\alpha\left(1+\alpha^2+2\lambda\left(-\lambda+\sqrt{-1+\lambda^2}\right)\right)}{(1+\alpha(\alpha-2\lambda))\sqrt{-1+\lambda^2}\,(1+\alpha(\alpha+2\lambda))} + \frac{e^{t\left(-\lambda+\sqrt{-1+\lambda^2}\right)} f0\,\alpha\left(1+\alpha^2-2\lambda\left(\lambda+\sqrt{-1+\lambda^2}\right)\right)}{(1+\alpha(\alpha-2\lambda))\sqrt{-1+\lambda^2}\,(1+\alpha(\alpha+2\lambda))} & t \geq 0 \\ 0 & \text{True} \end{cases}$$

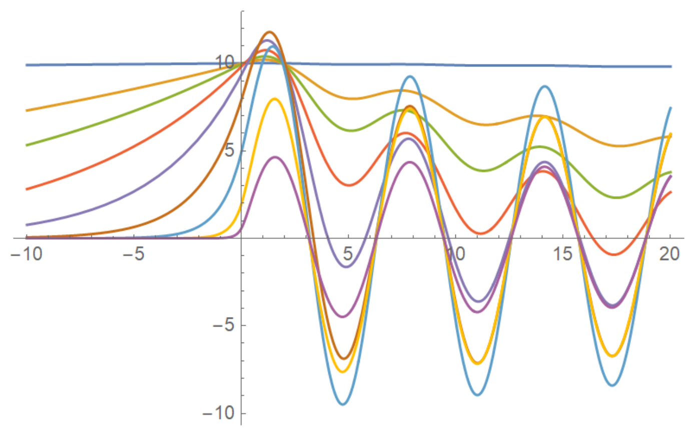
The first plot is overdamped motion as $\lambda > \omega_0$. The second plot is underdamped motion as $\lambda < \omega_0$.

2.

```
Plot[{X[t] /. {ω0 → 1, λ → 1.01, f0 → 10, α → 1/1000}, X[t] /. {ω0 → 1, λ → 1.01, f0 → 10, α → 1/32},
  X[t] /. {ω0 → 1, λ → 1.01, f0 → 10, α → 1/16}, X[t] /. {ω0 → 1, λ → 1.01, f0 → 10, α → 1/8},
  X[t] /. {ω0 → 1, λ → 1.01, f0 → 10, α → 1/4}, X[t] /. {ω0 → 1, λ → 1.01, f0 → 10, α → 1/2},
  X[t] /. {ω0 → 1, λ → 1.01, f0 → 10, α → 1}, X[t] /. {ω0 → 1, λ → 1.01, f0 → 10, α → 2},
  X[t] /. {ω0 → 1, λ → 1.01, f0 → 10, α → 4}}, {t, -10, 20}]
```



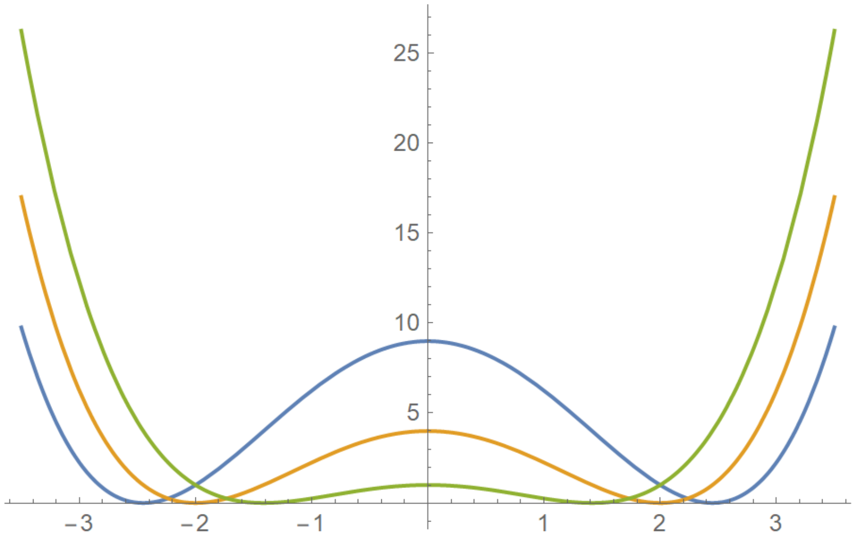

```
Plot[{X[t] /. {ω0 → 1, λ → 0.01, f0 → 10, α → 1/1000}, X[t] /. {ω0 → 1, λ → 0.01, f0 → 10, α → 1/32},
  X[t] /. {ω0 → 1, λ → 0.01, f0 → 10, α → 1/16}, X[t] /. {ω0 → 1, λ → 0.01, f0 → 10, α → 1/8},
  X[t] /. {ω0 → 1, λ → 0.01, f0 → 10, α → 1/4}, X[t] /. {ω0 → 1, λ → 0.01, f0 → 10, α → 1/2},
  X[t] /. {ω0 → 1, λ → 0.01, f0 → 10, α → 1}, X[t] /. {ω0 → 1, λ → 0.01, f0 → 10, α → 2},
  X[t] /. {ω0 → 1, λ → 0.01, f0 → 10, α → 4}}, {t, -10, 20}]
```



Problem 3

1.

```
$Assumptions = m > 0 && k > 0 && g > 0;
U[x_] = k^2/(4 g) - k/2 x^2 + g/4 x^4;
Plot[{U[x] /. {k → 6, g → 1}, U[x] /. {k → 4, g → 1}, U[x] /. {k → 2, g → 1}},
 {x, -3.5, 3.5}]
```



2.

$$\frac{\partial U}{\partial x} = 0$$

$$\implies x = 0, \ \pm\sqrt{\frac{k}{g}}$$

$$\frac{\partial^2 U}{\partial x^2} > 0$$

$$\implies x \neq 0$$

$$\implies x_{\text{stable}} = \pm\sqrt{\frac{k}{g}}$$

$$U(x_{\text{stable}}) = 0$$

```
Solve[U'[x] == 0, x]
```

$$\left\{\{x \to 0\}, \left\{x \to -\frac{\sqrt{k}}{\sqrt{g}}\right\}, \left\{x \to \frac{\sqrt{k}}{\sqrt{g}}\right\}\right\}$$

$$U''[x] \;/.\; \left\{\{x \to 0\}, \left\{x \to \sqrt{\frac{k}{g}}\right\}, \left\{x \to -\sqrt{\frac{k}{g}}\right\}\right\}$$

$$\{-k, 2\,k, 2\,k\}$$

$$U[x] \;/.\; \left\{\left\{x \to \sqrt{\frac{k}{g}}\right\}, \left\{x \to -\sqrt{\frac{k}{g}}\right\}\right\}$$

$$\{0, 0\}$$

3.

$$x(t) = \delta(t) + \sqrt{\frac{k}{g}}$$

$$\text{eom: } 0 = x''(t) + \frac{1}{m}\frac{\partial U}{\partial x}$$

$$0 = \delta''(x) + \frac{1}{m}\left(2k\,\delta(t) + 3\sqrt{gk}\,\delta(t)^2 + g\,\delta(t)^3\right)$$

```
EOM = m x''[t] + (U'[x] /. x → x[t])
```

$$-k\,x[t] + g\,x[t]^3 + m\,x''[t]$$

```
x[t_] = δ[t] + Sqrt[k/g];
```

```
EOM_EOM/m = EOM/m // FullSimplify // Expand
```

$$\frac{2\,k\,\delta[t]}{m} + \frac{3\,\sqrt{g\,k}\;\delta[t]^2}{m} + \frac{g\,\delta[t]^3}{m} + \delta''[t]$$

4.

$$\omega_0 = \sqrt{\frac{2k}{m}}$$

$$\epsilon_1 = \frac{3\sqrt{gk}}{m}$$

$$\epsilon_2 = \frac{g}{m}$$

$$\omega = A^2\left(-\frac{5{\epsilon_1}^2}{12{\omega_0}^3} + \frac{3\epsilon_2}{8\omega_0}\right)$$

$$\omega = -\frac{3A^2 g}{2\sqrt{2km}}$$

$$\frac{\omega}{\omega_0} = -\frac{3A^2 g}{4k}$$

$$\Omega_0 = \sqrt{\frac{2\,k}{m}};$$

$$\epsilon_1 = \frac{3\,\sqrt{g\,k}}{m};$$

$$\epsilon_2 = \frac{g}{m};$$

$$\omega = A^2\left(-\frac{5\,\epsilon_1{}^2}{12\,\Omega_0{}^3} + \frac{3\,\epsilon_2}{8\,\Omega_0}\right) \text{ // FullSimplify}$$

$$\frac{\omega}{\Omega_0} \text{ // FullSimplify}$$

$$-\frac{3\,A^2\,g}{2\,\sqrt{2}\,\sqrt{k\,m}}$$

$$-\frac{3\,A^2\,g}{4\,k}$$